

Chapter 1

Introduction to Vectors

1.1 Introduction

Mathematics is the fundamental tool to physics. Not only does it act as a tool in illustrating numerous physics ideas, it also helps physicists analyse systems of interests and experiments. Therefore, before we start to discuss Newtonian mechanics, we must first understand basic mathematical concepts that will be used extensively in the following text. Vector analysis is a mostly novel idea for new physics students. In the this introductory chapter, we are going to discuss this extensively.

1.2 Definition of a vector and a scalar

Vectors and scalars are the two mathematical descriptions of physical quantities, such as temperature, velocity and energy. The definitions of them will be stated as below.

1.2.1 Definition of a scalar

A *scalar* is a physical quantity which has magnitude only. Magnitude is the numerical value of the quantity, usually expressed in terms of a real number. Examples of scalars include energy, temperature and mass.

1.2.2 Definition of a vector

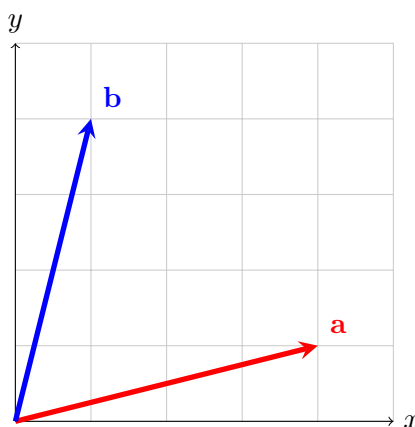
A *vector* is a physical quantity which has both magnitude and direction. The direction of the vector is defined by the set of coordinates, in other words, depending on the coordinate system, the direction of such vectors may be changed. Therefore, there exist another definition of vectors: for which their value (of direction) will undergo a change under coordinate transformation. Magnitudes of vectors never change in coordinate transformations. Examples of vectors include velocity and force.

In the following text, vectors will be represented in bold italic type. For example, \mathbf{v} is a vector. On the other hand, scalars will be simply denoted in plain italic type. Other common ways of denoting a vector includes putting a right arrow on top of it (as in \vec{v}) or underling it (as in \underline{v}).

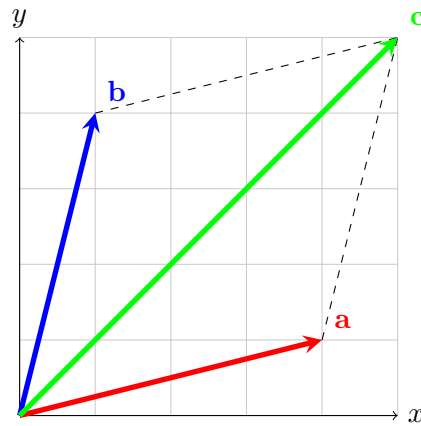
1.3 Vector Operations - Graphical

1.3.1 Addition of vectors

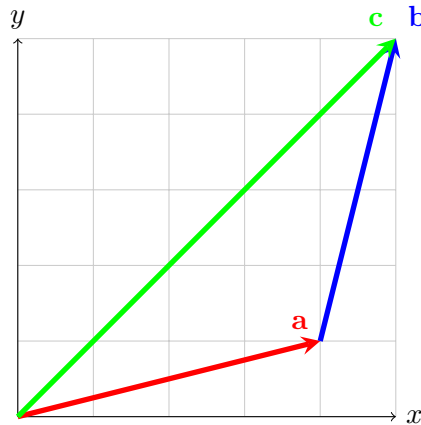
Vectors can be added to another, just like scalars. This can be done both numerically and graphically. There are two methods to add two vectors graphically. Consider vectors \mathbf{a} and \mathbf{b} .



Parallelogram method The first method is called parallelogram method. A vector is made up of two parts, the tip (pointy end) and the tail. The parallelogram method involves drawing a parallelogram from two vectors joined together with their tail ends. Draw the opposite parallel lines for each vector, then directly draw a vector from the tail ends of the vector to the opposite point of the parallelogram. The vector drawn is the sum of the two vectors. If $\mathbf{c} = \mathbf{a} + \mathbf{b}$, then

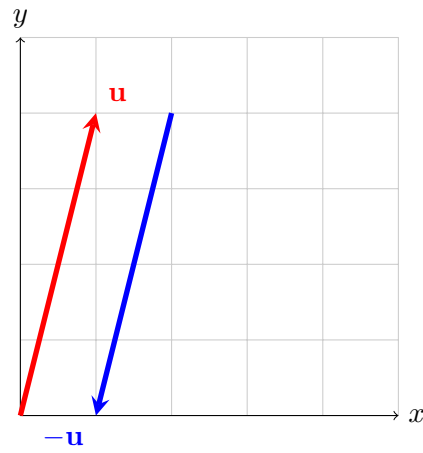


Tip-to-tail method The second method is called the tip-to-tail method. First move one of the vectors (**b**) until one tail end of a vector (**b**) is touching the tip of the other vector (**a**). Then directly draw a vector from the end of the first vector (**a**) to the tip of the other vector (**b**). Here $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is expressed:

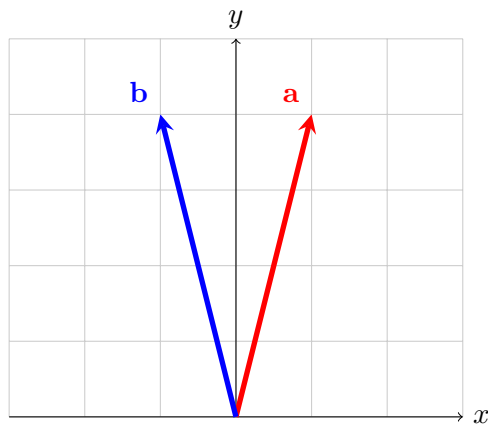


1.3.2 Subtraction of vectors graphically

As stated before, 2D vectors can be thought as an arrow on a plane. When the magnitude of a vector is negative, we can consider it to be a vector pointing to the opposite direction while having the same magnitude.

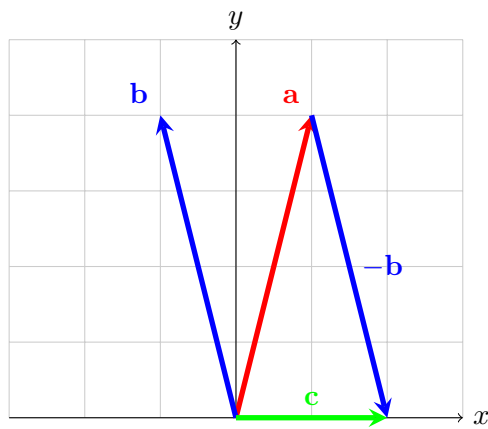


Therefore, subtraction of vectors can still be regarded as addition of vectors. Consider vectors \mathbf{a} and \mathbf{b} ,



By the results above,

$$\mathbf{c} = \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$



1.4 Unit vectors

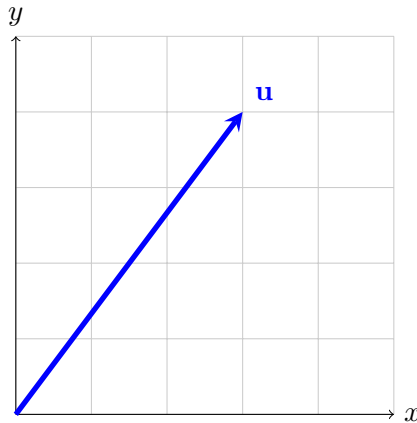
Unit vectors are vectors that have a magnitude of 1. They denote the direction of a family of vectors. It is formally defined as:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} \quad (1.1)$$

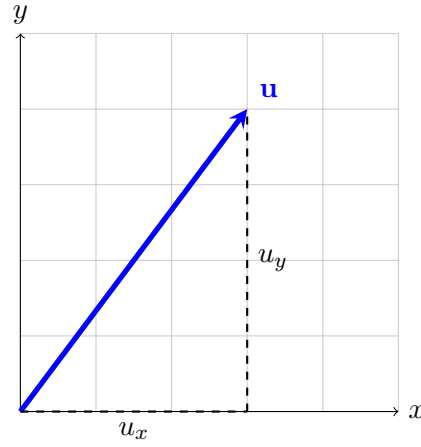
where $|\mathbf{u}|$ is the magnitude of the vector, also representable by u . Unit vectors are really useful in denoting the direction of vectors. In three-dimensional cartesian coordinate system, \hat{i} , \hat{j} and \hat{k} denote the x, y and z directions respectively. This implies that the vector $\mathbf{v} = 5\hat{i}$ is a vector having a magnitude of 5 units and pointing at the x-axis direction. Further unit vectors will be defined when they are required in text.

1.5 Components of vectors

Vectors can be thought as an arrow in a three-dimensional space. A n-component of a vector is the cosine of a vector with respect to a particular axis defined by the coordinate. We will demonstrate this in a two-dimensional plane. Consider a vector \mathbf{u} that is drawn from (0,0) to (3,4).



The components of this vector \mathbf{u} can be found by first drawing a perpendicular line of the x-axis to the tip of the vector. The components of the vector with respect to each axis can be measured easily by the lengths of the adjacent side and opposite side of the right triangle. Therefore, the components of \mathbf{u} are:



Therefore, by considering the right-angled triangle, the x and y components of vector \mathbf{u} are:

$$u_x = 5 \cos \theta = 3$$

$$u_y = 5 \sin \theta = 4$$

The vector \mathbf{u} , hence, can be thought as the addition of two vectors that are perpendicular to each other:

$$\mathbf{u} = 3\hat{i} + 4\hat{j}$$

or:

$$\mathbf{u} = u_x\hat{i} + u_y\hat{j} \quad (1.2)$$

Alternate form:

$$\mathbf{u} = u \cos \theta \hat{i} + u \sin \theta \hat{j} \quad (1.3)$$

This is the generalized representation of a 2D vector.

Three dimensional generalised representation of vectors In a three dimensional cartesian coordinate plane, a vector that extends from (x_1, x_2, x_3) to (x_1', x_2', x_3') can be expressed as:

$$\mathbf{a} = (x_1' - x_1)\hat{i} + (x_2' - x_2)\hat{j} + (x_3' - x_3)\hat{k}$$

$$\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad (1.4)$$

Therefore the components of a 3D vector:

$$\begin{cases} a_x = x_1' - x_1 \\ a_y = x_2' - x_2 \\ a_z = x_3' - x_3 \end{cases}$$

1.6 Vector Operations - Numerical

This section will illustrate the vector operations in component forms in a three-dimensional Cartesian coordinate system.

In the following text, we will use vectors \mathbf{a} , \mathbf{b} and \mathbf{c} as examples. They are defined as arbitrary vectors in a 3D Cartesian coordinate system.

$$\begin{cases} \mathbf{a} = a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}} + a_z \hat{\mathbf{k}} \\ \mathbf{b} = b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}} + b_z \hat{\mathbf{k}} \\ \mathbf{c} = c_x \hat{\mathbf{i}} + c_y \hat{\mathbf{j}} + c_z \hat{\mathbf{k}} \end{cases}$$

1.6.1 Addition and Subtraction of Vectors

Consider vectors \mathbf{a} and \mathbf{b} . The addition and subtraction of any vectors are illustrated as followed:

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}} + (a_z + b_z)\hat{\mathbf{k}} \quad (1.5)$$

$$\mathbf{a} - \mathbf{b} = (a_x - b_x)\hat{\mathbf{i}} + (a_y - b_y)\hat{\mathbf{j}} + (a_z - b_z)\hat{\mathbf{k}} \quad (1.6)$$

Addition of vectors is both commutative and associative. i.e.:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad (1.7)$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \quad (1.8)$$

1.6.2 Multiplying a vector with a scalar

Consider a vector \mathbf{a} and a constant k . To multiply the scalar with the vector, it is as simple as expressing it the way:

$$k * \mathbf{a} = k\mathbf{a} \quad (1.9)$$

When $k = 0$, the vector \mathbf{a} is a zero vector. It is a vector with zero magnitude and an arbitrary direction. Notice that the zero vector $\mathbf{0}$ is still a vector and possess vector properties.

1.6.3 Dot Product of Vectors

There are two kinds of multiplication for vectors. The first kind is named dot product. It is defined as below:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta \quad (1.10)$$

where a and b each denotes the magnitude of vectors \mathbf{a} and \mathbf{b} , and θ denotes the enclosed angle between the two vectors. In a 3D Cartesian coordinate system, the dot product can be simply illustrated as:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \quad (1.11)$$

Geometrically, the dot product of two vectors represent the product of a vector's magnitude and the other vector's component along the original arbitrarily set vector. One of the major uses of dot product in Newtonian Mechanics is the generalisation of work by a force, as illustrated later in Chapter 5.

1.6.4 Cross Product of Vectors

The second kind of vector multiplication is called cross product. While a dot product generates a scalar result, a cross product gives a vector. It is defined as follows:

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}} \quad (1.12)$$

where a and b each denotes the magnitude of vectors \mathbf{a} and \mathbf{b} , θ denotes the enclosed angle between the two vectors and the vector $\hat{\mathbf{n}}$ is a unit vector denoting the direction of the resultant vector. This direction can be found simply using the right-hand rule. In a 3D Cartesian coordinate system, the cross product can be simply illustrated as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (1.13)$$

The product vector can be easily obtained by solving the determinant. Although the algebraic solutions may be difficult, one would usually only encounter 3 by 3 matrices in Newtonian Mechanics, which hence can be easily solved. Geometrically, the cross product of two vectors represent the parallelogram formed using the two vectors, as one uses to geometrically add the two vectors together. The magnitude of the resultant vector is the area of the parallelogram, while the direction of the vector is perpendicular to the plane where the parallelogram mentioned is formed. (You might notice that for each plane in a 3D space, two vectors, that are mutually opposite to each other can be defined from the above definition. Indeed, the cross product of two vectors can result in either of the directions, but the direction is ultimately defined by the order of how the two vectors are multiplied since cross products are not *communitive*. Cross vectors will be seen being used numerous times in this text, one of which is in the definition of *torque*.

Chapter 2

Vector Analysis

Vector analysis is a very important concept in physics. This chapter intends to give a more detailed approach in the use of vectors in physics.

2.1 Geometrical Vector Analysis

Vectors are extremely useful in representing geometrical objects in a 3D space effectively. The use of different vector multiplication techniques allows us to analyse lines and planes in 3D space with ease.

2.1.1 Equation of a line

Recall that an equation of a line, in general, is represented by the equation:

$$y = mx + c \quad (2.1)$$

where m represents the slope of the line and c is the y-intercept of the line. Now we can reconstruct the line equation using vectors. In the following text, the notation \mathbf{r} will denote the position vector of a point from the origin. Consider \mathbf{l} be a vector pointing along the direction of the line. Vectors of any length parallel to the line can be represented by $\lambda\hat{\mathbf{l}}$, where λ is a constant. Suppose \mathbf{a} is a position vector of a point on the line. The equation of the line can therefore be represented as:

$$\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{l}} \quad (2.2)$$

Take the vector product of $\hat{\mathbf{l}}$ on both sides. Since the cross product of \mathbf{l} and \mathbf{l} is 0, the equation of the line can be represented as:

$$\mathbf{r} \times \hat{\mathbf{l}} = \mathbf{a} \times \hat{\mathbf{l}} = \mathbf{b} \quad (2.3)$$

where we have efficiently expressed the equation of a line with the use of cross products. This vector multiplication form is especially useful in finding the closest distance between a point and a line.

2.1.2 Equation of a plane

Vector equations can also represent the equation of a plane with ease. Recall that an area on a plane can be represented by a vector with a magnitude that is equal to the area and a direction normal to the plane, which is defined by a set of notations (e.g. right-hand rule). Now we will introduce a general rule for representing vectors on a plane. To define any point on a plane, we use two non-zero, non-parallel vectors:

$$\mathbf{r}' = \lambda \mathbf{b} + \mu \mathbf{c} \quad (2.4)$$

This linear combination of the two non-parallel vectors can express any point on a plane. Now suppose the plane is elevated in a 3D space by a vector \mathbf{a} :

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c} \quad (2.5)$$

where I have replaced the original displacement vector with a general displacement vector \mathbf{r} representing any point on the plane. If you are not happy with this definition, picture a 3D coordinate system in your mind. Define a plane of your choice to be the plane you are trying to describe. Now describe two non-parallel vectors on the plane. The linear combination of the two vectors should represent any point on the plane, *with respect to* a point of your choice on the plane. Define the displacement vector \mathbf{a} to be the vector of this arbitrary point. By similar arguments, any plane in the 3D space can be described by the above equation.

Now let us extend this general equation by vector operations. Define $\hat{\mathbf{n}}$ to be the unit vector normal to the plane. Since both \mathbf{b} and \mathbf{c} are vectors on the plane, they are perpendicular to the plane, which we can immediately deduce that the dot product of these vectors with the unit vector is 0. Hence, the equation of the plane can be simplified to the following:

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = d \quad (2.6)$$

where d is the perpendicular distance from the origin to the plane.

2.2 Triple Products

Triple products play an important role in the formation of the vector analysis theory. As an introductory text, I am going to briefly mention the definition, geometric meaning and usage of the triple products. You should consult your tutor or a book if you wish to find more extended proofs.

2.2.1 Scalar Triple Product

Suppose \mathbf{a} , \mathbf{b} and \mathbf{c} are non-zero and non-parallel vectors. The scalar triple product is formally defined as:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \quad (2.7)$$

The scalar triple product is an invariant with cyclic permutations. i.e.:

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned} \quad (2.8)$$

On the other hand, it requires a sign flip when anti-cyclic permutations are concerned:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] \quad (2.9)$$

In 3D cartesian coordinates, the scalar triple product can be easily expressed in terms of the components of the three vectors in a determinant form:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad (2.10)$$

You probably have noticed that the triple product outputs a scalar product, which is quite special in vector operations. In fact, the physical meaning of the scalar triple products is the volume of the parallelepiped formed by the three vectors when they all have the tip of their tails joined together. As the scalar triple products concerns a scalar and a vector product, we can always omit the brackets in writing as the vector product must be evaluated first. Otherwise, this would result in a scalar crossing a vector, which is not mathematically reasonable.

2.2.2 Vector Triple Product

Let us continue with our previous definitions of our example vectors. For the vectors \mathbf{a} , \mathbf{b} and \mathbf{c} , the vector triple product is defined as:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (2.11)$$

This can be proved using the 3D Cartesian Coordinate form of the triple product.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ (b_y c_z - b_z c_y) & (b_z c_x - b_x c_z) & (b_x c_y - b_y c_x) \end{vmatrix} \quad (2.12)$$

We may first concern the x component:

$$\begin{aligned} \hat{\mathbf{i}} \cdot [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] &= a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) \\ &= a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z \\ &= b_x(a_y c_y + a_z b_z) - c_x(a_y b_y + a_z b_z) + a_x b_x c_x - a_x b_x c_x \\ &= b_x(a_x c_x + a_y c_y + a_z b_z) - c_x(a_x b_x + a_y b_y + a_z b_z) \\ &= \hat{\mathbf{i}} \cdot [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \end{aligned}$$

Hence, the result can be repeated for $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ and the formula can be proved. Vector triple products have no general geometrical meaning and will not be extensively used in elementary physics.

2.2.3 Uses of triple products

Triple products, as mentioned are used extensively in space geometry and physics. An example with scalar triple products will be illustrated here. If we wish to prove that three vectors \mathbf{a} , \mathbf{b} and \mathbf{c} are co-planar, you may check if the scalar triple product of the three is equal to zero:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0 \quad (2.13)$$

Since three co-planar vectors cannot produce a parallelepiped, the volume of that will always be zero. The co-planarity of the vectors can therefore be checked with this simple result.

2.3 Orthogonal Basis

In this section, I will give a very brief introduction to the concept and use of orthogonal basis. More on the topic will be developed in subsequent chapters.

2.3.1 Definition of basis

A vector basis is a set of vectors in a vector space V defining every point in the vector space. The constituent vectors must be linearly independent and thus every vector in the vector space can be expressed as the linear combination of the set of vectors. In a 3D space, any three non-coplanar vectors \mathbf{a} , \mathbf{b} and \mathbf{c} constitute a basis which spans the entire 3D space.

2.3.2 Orthogonal Basis

For an orthogonal basis, the basis vectors are all mutually perpendicular to each other. They are further defined into right-handed or left-handed basis.

$$\begin{cases} [\mathbf{a}, \mathbf{b}, \mathbf{c}] > 0 & \text{iff right-handed} \\ [\mathbf{a}, \mathbf{b}, \mathbf{c}] < 0 & \text{iff left-handed} \end{cases} \quad (2.14)$$

where *iff* stands for *if and only if*. The orthogonal basis is further specialised to *right-handed orthonormal basis* where

$$|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1 \quad (2.15)$$

2.3.3 Cartesian Orthogonal Basis

Without further explanation, the concept of cartesian orthogonal basis should be pretty clear to you. It is clearly, from the definition of our previous section, an orthonormal basis. The properties of these vectors will be illustrated here briefly. Firstly, all of you should know that the basis vectors along the three axes x , y and z are denoted as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. Since it is orthonormal, it is evident that the following properties are true:

$$\begin{cases} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1 & \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1 & \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0 & \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 & \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0 \end{cases}$$

Also, since the basis is right-handed (by common definition):

$$\begin{cases} [\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}] = 1 \\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \\ \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} & \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} & \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \end{cases}$$

2.3.4 Reciprocal Basis

Reciprocal Basis is a unique set of vectors used to extract the components of vectors in a basis with vectors \mathbf{a} , \mathbf{b} and \mathbf{c} . The definition of reciprocal basis is as followed:

$$\begin{cases} \mathbf{a} \cdot \mathbf{A} = 1 & \mathbf{a} \cdot \mathbf{B} = 0 & \mathbf{a} \cdot \mathbf{C} = 0 \\ \mathbf{b} \cdot \mathbf{A} = 0 & \mathbf{b} \cdot \mathbf{B} = 1 & \mathbf{b} \cdot \mathbf{C} = 0 \\ \mathbf{c} \cdot \mathbf{A} = 0 & \mathbf{c} \cdot \mathbf{B} = 0 & \mathbf{c} \cdot \mathbf{C} = 1 \end{cases} \quad (2.16)$$

Therefore, the more common way to denote the definition of the reciprocal basis is:

$$\begin{cases} \mathbf{A} \equiv \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\ \mathbf{B} \equiv \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\ \mathbf{C} \equiv \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \end{cases} \quad (2.17)$$

Reciprocal vectors are very useful in finding the components of a vector in a basis. Suppose

$$\mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} \quad (2.18)$$

where λ , μ and ν are all constants. If we wish to find the a -component of \mathbf{r} , we can simply multiply the vector with the corresponding reciprocal vector:

$$\mathbf{A} \cdot \mathbf{r} = \lambda \mathbf{A} \cdot \mathbf{a} = \lambda$$

This allows us to directly find the component without further manipulation of the original vector.

