Simple Harmonic Motion

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Abstract

This is a document written specifically for students who has no previous knowledge of simple harmonic motion. It assumes from some elementary concepts in Newtonian Mechanics and the fundamental concepts of simple harmonic motion will be introduced. This includes the definition and some representations of simple harmonic motion using advanced mathematical concepts.

Contents

Preface and Usage of Notes

My last official physics notes for the Queen's College Physics Olympiad was made in 2016. Back then, it followed the format of my predecessor and had little explanation. We believed that the notes for QCPhO were made for revision, hence all the derivation should be omitted as that would prevent the students from realising the most important results of each topic. However, as I soon found out, it was not entirely desirable to leave out the derivations for the physical concepts as one may find it useful to go through that to further their understanding on the topic. I therefore added a supplementary section for the derivations of SHM at the back of my notes, yet it contained little content and was not self-sufficient to demonstrate the importance of SHM. This document is hence a follow-up of my previous set of notes and it aims to serve as a part of a set of reference notes for the coming tutors. It is also a fragment of the book that would (hopefully) be completed by the summer of 2019.

The content of the notes stretches far from what is required to compete in the Hong Kong Physics Olympiad. However, it is a part of the fundamental knowledge required to fully understand the mechanisms of oscillations and harmonic motion, which in turn is the foundation of much of modern physics developed in early 20th century. It is therefore most useful for the reader to go through the notes in their own pace, such that he or she can grasp the concepts fully before one moves on to move advanced topics.

I owe my gratitude to Dr. John Biggins and Dr. Neil Greenham of the University of Cambridge for the inspiration for this set of notes as their notes and courses on Oscillating Systems gave me new insight as to how the topic should be presented. I also owe my deep thanks to Albus Poon, Matthew Tong, Thomas Yuen and Hillman Lai for proofreading the set of notes for me.

1 Definition of Harmonic Motion

A harmonic motion is a type of *periodic motion* that repeats itself in a given set of period. This type of motion can be found anywhere from mechanical to electrical situations. It is also the fundamental concept in *wave optics*. Examples of harmonic motion include pendulums and alternate circuits.
Since harmonic motions are so ubiquitous, it is important that we analyse and understand the physical and mathematical formulations of it. In the following sections, we are exactly going to do that. As for the next section, various preliminary knowledge points are listed to refresh your memory before you begin reading the following sections.

2 Preliminary knowledge

2.1 Kinematics

2.1.1 Displacement

In a 3D space, we define an object's *displacement* as a *separation vector* between its initial position and its final position, both defined as a point in the space. This *separation vector* is called a *displacement vector*, with its tail originating from the initial point and its tip pointing to its final point. In the following text, the displacement of a mechanical system will be described using x or s, with its vector symbol dropped off for simplification.

2.1.2 Velocity

The *velocity* of an object is defined as the infinitesimal change of an object's displacement in a short period of time, i.e. the derivative of displacement with respect to time. This is represented by $\dot{x}$ in the following text.

2.1.3 Acceleration

The *acceleration* of an object is defined as the derivative of velocity with respect to time. Since velocity is the derivative of displacement with respect to time, the acceleration is therefore the double derivative of displacement with respect to time. This is represented by $\ddot{x}$ in the following text.

2.2 Newton's Laws of Motion

Newton's Laws of Motion are physical laws that are formulated from experimental evidence. They are the basis of the formation of *Newtonian Mechanics*.

2.2.1 Newton's First Law of Motion

Newton's First Law of Motion states that an object retains its form of motion if no net force (or, torque) is acting on it. It is alternatively named as *Law of Inertia*. You are reminded that this is not the same as *Inertia*, which is a physical property of an object that describes its reluctance to change its state of motion.

2.2.2 Newton's Second Law of Motion

The *momentum* of a body is defined as the mass times the velocity of a body. It is a special property of a moving body. In mathematical symbols:

$$\mathbf{p} = m\mathbf{v} \tag{1}$$

Newton's Second Law of Motion states that the rate of change of momentum of an object is proportional to the force acting on the object and has the same direction as the force. Notice that this "force" mentioned actually refers to the net force acting on the object. The mathematical statement is as follows:

$$\mathbf{F}_{net} = \frac{d\mathbf{p}}{dt} \tag{2}$$

It also has an alternative form that may be more familiar to you:

$$\mathbf{F}_{net} = m\mathbf{a} \tag{3}$$

2.2.3 Newton's Third Law of Motion

Newton's Third Law of Motion states that if one body, A, exerts a force, $\mathbf{F}_{A\to B}$, on another body, B, then the force, $\mathbf{F}_{B\to A}$ exerted by B on A is equal and opposite to the original force. Mathematically,

$$\mathbf{F}_{A\to B} = -\mathbf{F}_{B\to A} \tag{4}$$

Notice that the action and reaction act on different bodies.

2.3 Energy and Power

2.3.1 Work-Energy Theorem

The *Work-Energy Theorem* is a direct consequence of *Newton's Second Law of Motion*. Its formulation is as below:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

$$\begin{aligned}
\mathbf{F} \cdot d\mathbf{x} &= m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= \int_{\mathbf{u}}^{\mathbf{v}} m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= m\int_{\mathbf{u}}^{\mathbf{v}} \frac{d\mathbf{x}}{dt} \cdot d\mathbf{v} \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= m\int_{\mathbf{u}}^{\mathbf{v}} \mathbf{v} \cdot d\mathbf{v} \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= m\int_{u}^{v} v dv \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= m\frac{1}{2}v^2\Big|_u^v \\
\int_C \mathbf{F} \cdot d\mathbf{x} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \qquad (5)
\end{aligned}$$

The right-hand side of the equation represents the change of *kinetic energy* of the system. The left-hand side of the equation is the work done by an external force, where $\int_C$ corresponds to a path integral. This work done can be done by a *conservative force*, for which this term can then be replaced by a change of potential energy.

2.3.2 Power

Power is defined as the derivative of energy with respect to time. Mathematically,

$$P = \frac{dE}{dt} \qquad (6)$$

You can also regard *power* as the rate of doing work.

2.3.3 Conservation of Energy

Conservation of Energy is an independent physical law stating that the total energy of a system is conserved in an isolated system. Notice that this is not the same as the work-energy theorem. The work-energy theorem is a *direct consequence of the Second Law of Motion*, while conservation of energy is an independent physical law from experimental results. You should make sure that you never confuse between the two laws as they are fundamentally different, although you may obtain the result of energy conservation from the work-energy theorem.

2.4 Circular Motion

2.4.1 Analogy to Linear Motion

In a lot of ways, quantities in circular motion is directly analogous to that in linear motion. The following is a table of quantities used in circular motion.

Notations in linear and circular motion		
Quantity	Linear	Circular
Displacement	$\mathbf{x}$	θ
Velocity	$\dot{\mathbf{x}}$	$\dot{\theta}$
Acceleration	$\ddot{\mathbf{x}}$	$\ddot{\theta}$
Inertia	m	I
Force	$\mathbf{F}$	τ
Momentum	$\mathbf{p}$	$\mathbf{L}$

The angular equivalent of linear kinematic terms are named by adding "angular" before the original terms. Some important equations are given as follows:

$$v = r\dot{\theta} = r\omega \tag{7}$$

$$a = r\ddot{\theta} = r\alpha \tag{8}$$

where the vector signs are dropped for convenience. The angular equivalent of mass is termed *moment of inertia*. In *fixed axis rotation*, you can calculate a body's moment of inertia by the following formula:

$$I = \int r^2 dm \tag{9}$$

The angular equivalent of force and momentum is termed *torque* and *angular momentum*. The relationship between the linear terms and angular terms is given by:

$$\tau = \mathbf{r} \times \mathbf{F} \tag{10}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{11}$$

where the two are related by the rotational form of Second Law of Motion:

$$\tau = \frac{d\mathbf{L}}{dt} \tag{12}$$

where $\mathbf{r}$ is the displacement vector from the pivot. The total kinetic energy is, hence, given by:

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{13}$$

2.5 Frequency and Period

The *frequency* of a harmonic (circular) motion is the number of complete cycles achieved in one second. It is usually denoted by the symbol f or ν. The *period* of a motion is the time required for a complete cycle to be achieved, and it is denoted by the symbol T. It should be apparent that the reciprocal of period gives the frequency of the motion:

$$\nu = \frac{1}{T} \tag{14}$$

The angular frequency ω of a motion is the angle (in radians) achieved in one second. It is simply defined by:

$$\omega = 2\pi\nu = \frac{2\pi}{T} \tag{15}$$

You might be confused that ω both represents angular velocity and angular frequency. In fact, if you look at the definition, they are actually referring to the same thing!
If you do not understand the above notes made, you should consult your tutor, previous notes or any mechanics books as all of the concepts will be applied in the following sections.

3 Definition of Simple Harmonic Motion and the spring-mass system

A *simple harmonic motion* is a special kind of harmonic motion for which when the system is slightly disturbed from its equilibrium position, it would oscillate with a restoring force proportional to the displacement. This can be represented by a general equation:

$$\ddot{x} = -\omega^2 x \tag{16}$$

where ω is the angular frequency of the oscillation. This equation is often also called the fundamental SHM equation. We shall return to this later.
An example of a *simple harmonic oscillator*, a system that undergoes SHM, is the spring mass system. (See figure 1) Consider a mass m resting on a frictionless surface while connecting to the wall by a spring of spring constant k. When the mass is slightly displaced from the equilibrium position, by Hooke's Law, the restoring force is given by:

$$F_{restoring} = -kx \tag{17}$$

where the negative sign is present to signify that the direction of extension of the spring is opposite to the direction of the force. Combining this with Newton's Second Law of motion gives:

$$m\ddot{x} = -kx$$

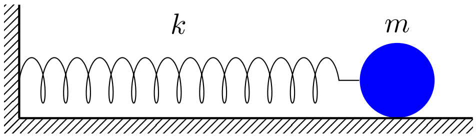

Figure 1: A simple harmonic system called a *spring-mass system.* A ball of mass m is connected to the wall by a spring with a spring constant k. Assume that all surfaces are frictionless.

$$\ddot{x} = -\frac{k}{m}x \tag{18}$$

The equation of motion of the spring mass system obeys with that of the general SHM motion. This shows that the system undergoes SHM.

4 Solutions of the general SHM equation

Let us return to the general equation of SHM and solve the differential equation. By substitution of the eigenfunction of differential equations, we have:

$$\ddot{x} + \omega^2 x = 0$$
$$\Rightarrow \rho^2 + \omega^2 = 0$$
$$\rho = \pm i\omega$$

Hence we will obtain:

$$x(t) = Ee^{i\omega t} + Fe^{-i\omega t} \tag{19}$$

At this point, you should notice that the solution of the differential equation must be real. Taking the real part of the solution, we have:

$$x(t) = C\cos\omega t + D\sin\omega t \tag{20}$$

where C, D, E and F are arbitrary constants. This therefore gives us the displacement of the system. You may check that our solution obeys the original differential equation by substituting the solutions in:

$$\dot{x} = -\omega(C\sin\omega t + D\cos\omega t) \tag{21}$$

$$\ddot{x} = -\omega^2(C\cos\omega t + D\sin\omega t) = -\omega^2 x \tag{22}$$

We can further rewrite the solution in a simpler form. By using the compound angle formula, and suppose that:

$$\begin{cases} C = A\cos\phi \\ D = -A\sin\phi \end{cases} \tag{23}$$

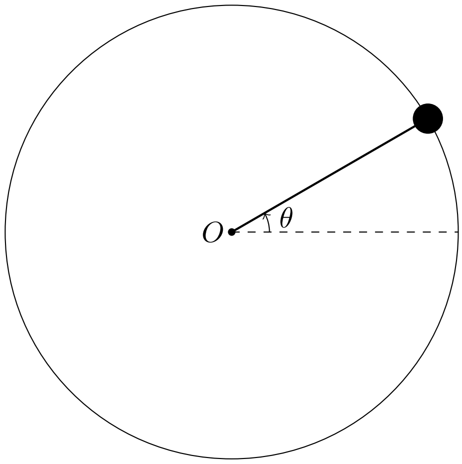

Figure 2: A ball undergoing circular motion around the origin. The angular displacement θ is measured from the horizontal x-axis.

Hence we have,

$$\begin{aligned} x(t) &= A\cos\phi\cos\omega t - A\sin\phi\sin\omega t \\ &= A\cos(\omega t + \phi) \end{aligned} \tag{24}$$

Now the solution gives a clear physical picture: it shows that the system would undergo a periodic motion as predicted by the time-varying trigonometric term. The amplitude of the motion is A, the angular frequency is given by ω, and ϕ is the phase constant. A and ϕ are arbitrary constants, which are to be determined by the initial conditions of the system. Now you should understand why we used ω^2 as our proportionality constant in the previous section: it is rooted when we solve the differential equation. This clearly shows the correlation of angular frequency of the motion to the state of motion of the system.

5 Relationship of SHM and circular motion

Let us now consider the circular motion of a ball (point mass) moving around the origin with a radius of A. We will denote the starting angle θ_0 of the ball as ϕ. At time t, the ball has moved an angle ωt, where ω is the angular velocity of the ball. The total angle displaced, measured from the positive x-axis direction, is $(\phi + \omega t)$. Let us try to project the ball's motion on one single axis: the x-axis. Since at any given time, the ball's displacement is given by $A\cos\theta$, we can now set-up an equation expressing the x-component of the ball's displacement with respect to time:

$$x(t) = A\cos\theta = A\cos(\omega t + \phi) \tag{25}$$

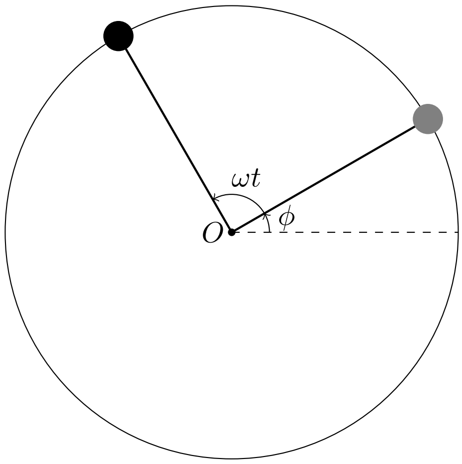

Figure 3: A ball undergoing circular motion around the origin with an angular velocity ω. The grey-coloured ball shows the initial position of the ball. The phase angle ϕ shows the initial angular displacement, which is also called the starting angle. At time t, the ball moved an angle ωt from its initial position, measured from the origin.

Voilà! We have obtained the solution to the fundamental differential equation of simple harmonic motion! By differentiation, we can also obtain the velocity and acceleration of the motion:

$$\dot{x}(t) = -\omega A \sin(\omega t + \phi) \tag{26}$$

$$\ddot{x}(t) = -\omega^2 A \cos(\omega t + \phi) \tag{27}$$

You may also consider the y-component of the displacement of the ball and obtain a similar trigonometric solution. This solution, however, is in sine form, but you might recall me using the trick of double angle formula when obtaining equation (24). By using a different substitution of the arbitrary constant, you can obtain the sine solution of the differential equation (3).
We have therefore obtained the result that SHM is the linear projection of a circular motion. This circle is called the *reference circle* and is extremely useful when we deal with the complex representation of SHM.

6 Examples of Simple Harmonic Motion

As I have mentioned, simple harmonic motion is ubiquitous. Here I have listed some common SHM examples:

6.1 Simple Pendulum

A *simple pendulum* is made of a point mass connected to an inextensible massless string. The conventional symbol for the length of spring is l. There

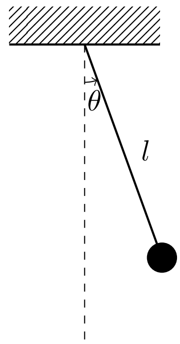

Figure 4: A simple pendulum set up. It consists of a bob hanging from the ceiling by an inextensible massless string. The angle θ should be small.

are two ways to consider the motion of a simple pendulum: Firstly, we may try and consider the horizontal displacement of a knob. From Newton's Second Law,

$$\begin{cases} T\cos\theta = mg \\ T\sin\theta = m\ddot{x} \end{cases}$$

where T is the tension in the spring. Hence, you have,

$$\ddot{x} = -g\tan\theta$$

where the negative sign is added as the force direction is opposite to the direction of the displacement. For small angles, $\theta \approx \sin\theta \approx \tan\theta$, therefore:

$$\ddot{x} = -g\sin\theta = -\frac{g}{l}x \tag{28}$$

Hence we have:

$$\omega^2 = \frac{g}{l} \tag{29}$$

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{30}$$

The second method concerns the analysis of the rotational motion of the pendulum about the pivot (where the string attaches to the roof/ceiling). By the rotational form of Newton's Second Law of Motion,

$$\tau = I\ddot{\theta} \tag{31}$$

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

Again, we have to use the small angle approximation,

$$ml^2\ddot{\theta} = -mgl\theta$$

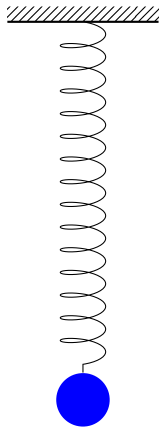

Figure 5: A vertical spring-mass system. Simply a rotated version of the original spring-mass system presented in Figure 1.

Hence we have,

$$\ddot{\theta} = -\frac{g}{l}\theta \tag{32}$$

We hence obtain the same result as above.

6.2 Torsional Pendulum

A *Torsional Pendulum* is analogous to a linear spring-mass system. It is made up of a rotational wire with stiffness C by the restoring couple connected to an arbitrary rotating body. By analysing the rotational motion using equation (31), we obtain,

$$\tau = I\ddot{\theta} = -C\theta$$

$$\ddot{\theta} = -\frac{C}{I}\theta \tag{33}$$

6.3 Vertical Spring-mass System

The *vertical spring-mass system* is a bit trickier due to the constant forcing fo weight. (See Figure 5.) However, the analysis is similar. By Newton's Second Law, we have:

$$m\ddot{x}_1 = -mg - kx_1 \tag{34}$$

Now suppose $-kx = -mg - kx_1$, hence $\ddot{x} = \ddot{x}_1$:

$$\Rightarrow m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x \tag{35}$$

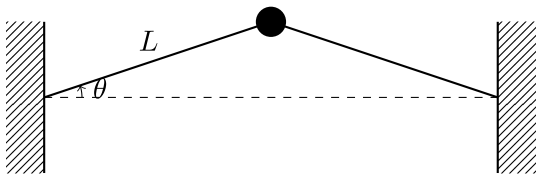

Figure 6: A ball of mass m put at the centre of a string of a length of L. The string is under constant tension T. The displaced angle θ is assumed to be small.

Hence the angular frequency and period of SHM is still the same as the horizontal spring-mass system

$$\omega^2 = \frac{k}{m} \tag{36}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{37}$$

6.4 Mass at centre of string under constant tension T

This system is self-explained in Figure 6. It looks like an arbitrary mechanical problem but SHM is still apparent here. By considering vertical forces:

$$m\ddot{x} = -2T\sin\theta$$

Using small angle approximation (again), we have:

$$\ddot{x} = -\frac{2T}{m}\tan\theta$$

$$\ddot{x} = -\frac{2T}{mL}x \tag{38}$$

Hence we have:

$$\omega^2 = \frac{2T}{mL} \tag{39}$$

$$T = 2\pi\sqrt{\frac{mL}{2T}} \tag{40}$$

6.5 Fixed length non-viscous liquid in a U-tube of constant cross section

We will now move on to something that you may not be familiar with. Don't worry, the only thing here to recall from is the *buoyant force*:

$$B = \rho g V \tag{41}$$

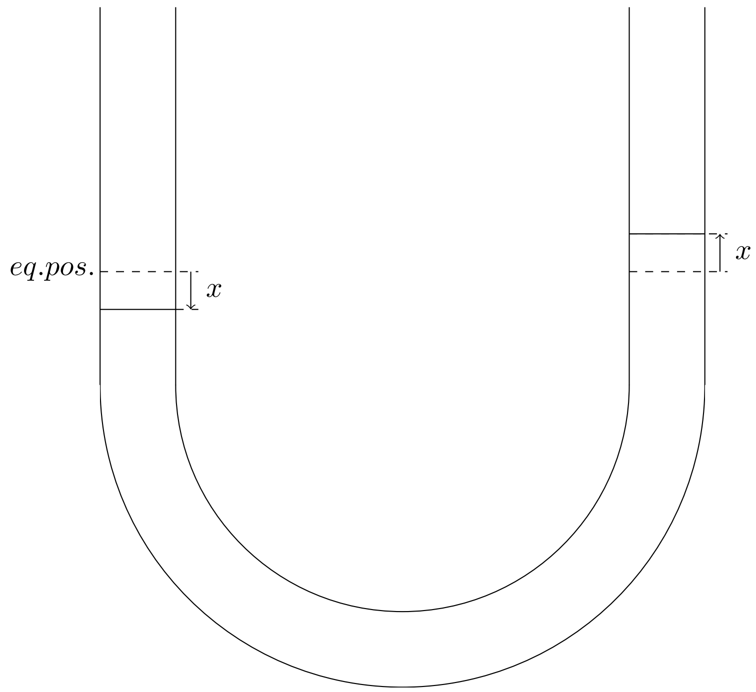

Figure 7: A schematic diagram for a U-tube. This diagram shows the snapshot of the oscillating system when the liquid has zero velocity, i.e. maximum displacement from the equilibrium position. The maximum difference in the height of the two hands of the tube is $2x$.

which is actually from the Archimedes' Principle stating that the force the applied to an object in a fluid is given by the mass of fluid displaced. A *U-tube* is a device commonly used in some chemical reactions (primarily for letting air to pass through, putting drying agent would help dry the air). Therefore, by considering the movement of a small unit of liquid in the tube with height x, we have:

$$\rho A l \ddot{x} = -2\rho x A g \tag{42}$$

where the left-hand side of the equation is the term from Newton's Second Law of motion of the small volume of fluid moved, and the right-hand side is the buoyant force acting on the small volume of fluid. (Consider the equilibrium position and the subsequent motion of the body of liquid - you will realise that only one half of the labeled liquid actually move; you can consider that part of the liquid to be the only part that is in motion.) We therefore have:

$$\ddot{x} = -\frac{2g}{l}x \tag{43}$$

Hence we have:

$$\omega^2 = \frac{2g}{l} \tag{44}$$

$$T = 2\pi\sqrt{\frac{l}{2g}} \tag{45}$$

6.6 Hydrometer in liquid with density ρ

A *hydrometer* is a device that is used to measure the density of a fluid by the depth it sinks into the liquid. It can be set into SHM if we push it slightly downwards after it reaches the equilibrium position floating in a tank of liquid, water for example. In this, we only have to consider the buoyant force (which is the restoring force) on the hydrometer due to small displacement of the it on the surface of water:

$$m\ddot{x} = -\rho g A x$$

$$\ddot{x} = -\frac{\rho g A}{m}x \tag{46}$$

Hence we have:

$$\omega^2 = \frac{\rho g A}{m} \tag{47}$$

$$T = 2\pi\sqrt{\frac{m}{\rho g A}} \tag{48}$$

As stated, there are too many examples of SHM in our world for me to include all of them in this document. You are therefore highly encouraged to explore more systems that undergo SHM.

7 Energy and Power in SHM

7.1 Energy Conservation in SHM

When we analyse the systems undergoing SHM, it is also important to consider the energy of the system. The fact that the velocity goes to zero at the extreme points and that it reaches the maximum at the equilibrium position raises a very important idea about the exchange of energy between the potential and kinetic energy. Let us start by considering the spring-mass system:

$$\ddot{x} = -\frac{k}{m}x \tag{18}$$

$$x(t) = A\cos(\omega t + \phi) \tag{25}$$

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi) \tag{26}$$

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi) \tag{27}$$

where $\omega = \sqrt{\frac{k}{m}}$. By considering the elastic potential energy and kinetic energy of the system, we have:

$$PE = \frac{1}{2}kx(t)^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi) \tag{49}$$

$$KE = \frac{1}{2}m\dot{x}(t)^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \tag{50}$$

By rewriting $k = m\omega^2$, we have,

$$\begin{aligned} E_{total} &= PE + KE \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2 \end{aligned} \tag{51}$$

Since both the amplitude (set by initial conditions) and the spring constant are non time-varying, the total energy is constant. Energy is conserved in SHM. It is converted between PE and KE. Since the total mechanical energy is conserved, the power of the system, by taking the derivative of a constant term, is zero.

7.2 Energy-method to obtain fundamental equation of SHM

With this result, we now are able to obtain an expression using a so-called "energy-method". Let us return to the simple pendulum. To use the *energy-method*, we first write the expressions of energy in terms of time-varying constants:

$$\begin{aligned} E_{total} &= PE + KE \\ &= mgl(1 - \cos\theta) + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}mgl\theta^2 + \frac{1}{2}ml^2\dot{\theta}^2 \end{aligned} \tag{52}$$

where I have used $\cos\theta = 1 - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4)$. By realising that the overall power is zero, we have:

$$\dot{E} = mgl\theta\dot{\theta} + ml^2\dot{\theta}\ddot{\theta} = 0 \tag{53}$$

Since $\dot{\theta} = 0$ is the trivial solution, it can be ignored. By cancelling $\dot{\theta}$ from both sides, we have,

$$\begin{aligned} -mgl\theta &= ml^2\ddot{\theta} \\ \ddot{\theta} &= -\frac{g}{l}\theta \end{aligned} \tag{32}$$

which is exactly as the fundamental equation obtained before.

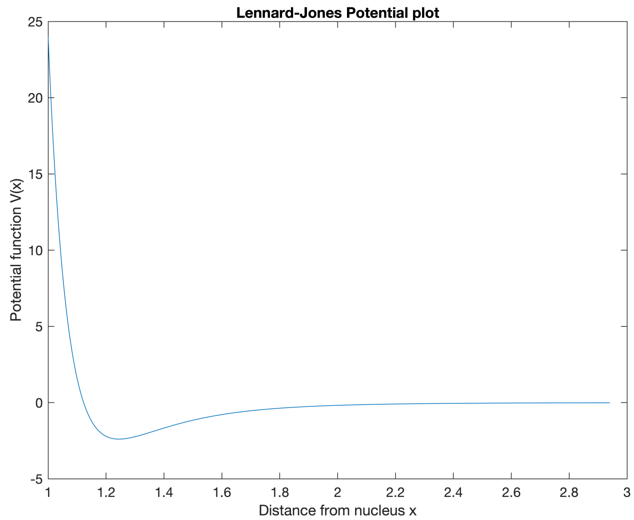

Figure 8: An example of a potential function $V(x)$. This potential function is called *Lennard-Jones Potential*, and is fundamental to the theory of orbitals in chemistry as it models the potential energy around the nucleus of an atom. The constants are arbitrarily set.

7.3 Harmonic motion in a general potential well

You might wonder why we can employ such methods in finding the equation of motion of a body. It is not very obvious to find a general potential energy that is quadratic, for which the fundamental solution of SHM can then be obtained. However, it is in fact true for all potential energies to have a quadratic form which are in potential wells.

A *potential well* is a region in space where the potential is significantly lower than the surrounding values. The particle is "trapped" inside the well if it does not have enough energy to escape the well. You might imagine a plot of $V(x)$ against x as in Figure 8, where $V(x_0)$ is the local minima. Let us try to find the expression for potential energy for a small disturbance δx around the equilibrium point x_0. Using *Taylor's expansion*:

$$\begin{aligned} V(x_0 + \delta x) &= V(x_0) + (\delta x)\frac{dV}{dx}\bigg|_{x=x_0} + \frac{1}{2}(\delta x)^2\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \frac{1}{6}(\delta x)^3\frac{d^3V}{dx^3}\bigg|_{x=x_0} + \dots \\ &= V(x_0) + (\delta x)\frac{dV}{dx}\bigg|_{x=x_0} + \frac{1}{2}(\delta x)^2\frac{d^2V}{dx^2}\bigg|_{x=x_0} + \mathcal{O}((\delta x)^3) \end{aligned} \tag{54}$$

where I have neglected the higher order terms (expressed by the big O notation). Since $\frac{dV}{dx}$ at that point is zero by definition (local minima), we

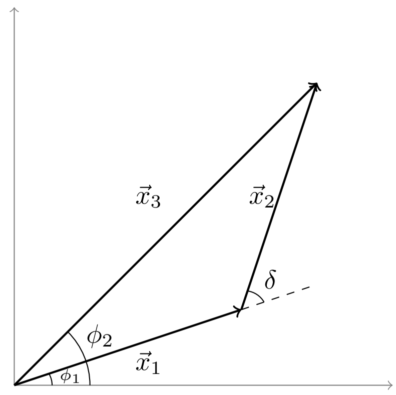

Figure 9: A vector diagram showing the addition of two solutions in vector form. The angle δ is the angle between the two phase difference angles ϕ_1 and ϕ_2. The vectors can be regarded as phasors (to be mentioned later).

have:

$$V(x_0 + \delta x) = V(x_0) + \frac{1}{2}(\delta x)^2 \frac{d^2 V}{dx^2}\Big|_{x=x_0} \tag{55}$$

Hence,

$$\Delta V = \frac{1}{2}(\delta x)^2 \frac{d^2 V}{dx^2}\Big|_{x=x_0} \tag{56}$$

We have hence obtained that the potential change is quadratic. Since the potential energy of the system is the potential multiplied by a non-time-varying physical quantity, we have found out that the potential energy change is also quadratic. Therefore, we can conclude that a small disturbance of a system in a potential well would undergo SHM.

8 Superposition of SHM

8.1 Vibration having equal frequencies in one dimension

Suppose we have two solutions, $x_1(t)$ and $x_2(t)$ to an SHM equation. $x_3(t) = x_1(t) + x_2(t)$, in principle, is also a solution to the equation. Therefore, to find the resulting motion of a system under simultaneous effect of two harmonic oscillations of equal frequencies but different amplitude and phases, we can represent each SHM as a vector and carry out vector addition: where

$$\begin{cases} x_1(t) = A_1 \cos(\omega t + \phi_1) \\ x_2(t) = A_2 \cos(\omega t + \phi_2) \end{cases} \tag{57}$$

To find the amplitude of the motion, from Figure 9, we have:

$$\begin{aligned} R^2 &= (A_1 + A_2\cos(\phi_2 - \phi_1))^2 + (A_2\sin(\phi_2 - \phi_1))^2 \\ &= A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1) \\ &= A_1^2 + A_2^2 + 2A_1A_2\cos\delta \end{aligned} \tag{58}$$

where $\delta = \phi_2 - \phi_1$. The new phase constant is given by:

$$\tan\psi = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} \tag{59}$$

The resultant SHM is hence given by:

$$x_3(t) = R\cos(\omega t + \psi) \tag{60}$$

which is an SHM with an amplitude of R and a phase constant of ψ.

8.2 Vibrations having different frequencies in one dimension

Suppose now we have two harmonic oscillators with the same amplitude but different frequency simultaneously effecting a motion (phase constant assumed zero for simplification),

$$\begin{cases} x_1(t) = A\cos(\omega_1 t) \\ x_2(t) = A\cos(\omega_2 t) \end{cases} \tag{61}$$

By superposition, we have,

$$\begin{aligned} x_3(t) &= x_1(t) + x_2(t) \\ &= A(\cos(\omega_1 t) + \cos(\omega_2 t)) \\ &= 2A\cos(\frac{\omega_1 + \omega_2}{2}t)\cos(\frac{\omega_1 - \omega_2}{2}t) \end{aligned} \tag{62}$$

The resultant motion is represented by Figure 10. which is an oscillation with a fast frequency and a slowly oscillating amplitude. The slower frequency determines the amplitude of the oscillation we can detect, which is given by the angular frequency $(\omega_2 - \omega_1)$, as the absolute value of cosine reaches maxima twice the rate it reaches maximum in its original function (as we are talking about the amplitude here). This effect is called *beats*, and is used extensive in tuning instruments by musicians.

8.3 Vibrations having the same frequency in two dimensions

Now we switch our focus to the superposition of vibrations in two dimensions. Suppose we have a set-up as followed:
and let us suppose the initial position of the ball is arbitrary but small

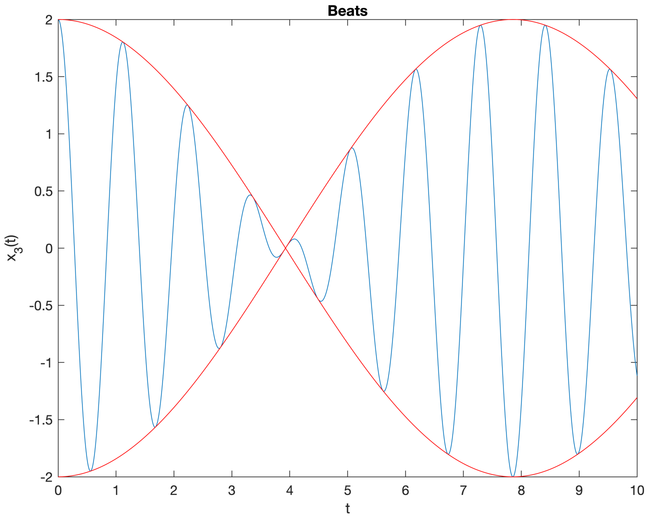

Figure 10: A plot of the solution x_3 (blue line). For simplification, the arbitrary constants are substituted by chosen values. The red lines represent the change of the overall amplitude of the oscillation. It vibrates twice as fast as the function as only the *amplitude* of the trigonometric function is concerned.

enough to ensure SHM. Since SHM occurs independently on the two axes, we may have the solutions:

$$\begin{cases} x = a_1 \sin(\omega t + \phi_1) \\ y = a_2 \sin(\omega t + \phi_2) \end{cases} \tag{63}$$

for which we can rewrite the solutions as:

$$\begin{cases} \frac{x}{a_1} = \sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1 \\ \frac{y}{a_2} = \sin \omega t \cos \phi_2 + \cos \omega t \sin \phi_2 \end{cases} \tag{64}$$

By eliminating ωt,

$$(\frac{x}{a_1} \sin \phi_2 - \frac{y}{a_2} \sin \phi_1)^2 + (\frac{y}{a_2} \cos \phi_1 - \frac{x}{a_1} \cos \phi_2)^2 = \sin^2(\phi_2 - \phi_1)$$

$$(\frac{x}{a_1})^2 + (\frac{y}{a_2})^2 - 2(\frac{x}{a_1})(\frac{y}{a_2}) \cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1) \tag{65}$$

which is the general equation of an inclined ellipse. By varying the phase difference, we would obtain different modes of vibration.
For example, when $\phi_2 - \phi_1 = \frac{\pi}{2}$, we have:

$$(\frac{x}{a_1})^2 + (\frac{y}{a_2})^2 = 1 \tag{66}$$

which is a general form of ellipse, with semi-major axis a_1 and semi-minor axis a_2. Of course, if $a_1 = a_2$, this becomes a circle.
When $\phi_2 - \phi_1 = 2\pi n$, where $n \in \mathbb{N} \cup \{0\}$, we have:

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} \mp \frac{2xy}{a_1 a_2} = 0 \tag{67}$$

where the sign is minus when n is even and plus when n is odd. Hence,

$$\Rightarrow xa_1 \mp ya_2 = 0$$

$$y = \pm\frac{a_2}{a_1}x \tag{68}$$

which is a straight line with a slope of $\pm\frac{a_2}{a_1}$, depending on n. These vibrations can be easily demonstrated on an oscilloscope.

8.4 Vibrations having different frequencies in two dimensions

When the two superposing vibrations have different frequencies, things get really complicated. These vibrations are usually represented by figures called *Lissajous figures*, which are mathematically challenging and invented to specifically analyse these vibrations. We will not go in depth here.

9 Complex representation of SHM

Let us return to the discussion of relationships between SHM and circular motion. Let us consider the displacement, velocity and acceleration of the solution of SHM:

$$x(t) = A\cos(\omega t + \phi) \tag{25}$$

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi) \tag{26}$$

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi) \tag{27}$$

where ω is the angular frequency determined by the system of interest. It is obvious that we can rewrite the velocity and acceleration expressions:

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi) = \omega A\cos(\omega t + \phi + \frac{\pi}{2}) \tag{69}$$

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi) = \omega^2 A\cos(\omega t + \phi + \pi) \tag{70}$$

The velocity and acceleration is a phase shift of the displacement by $\frac{\pi}{2}$ and π respectively (with their amplitude multiplied by ω and ω^2, of course), we can represent this result using a diagram: Figure 11 is called a phasor diagram. This can easily simplify our analysis as all these phasors rotate with the same frequency and hence allow us to visualise the phase difference

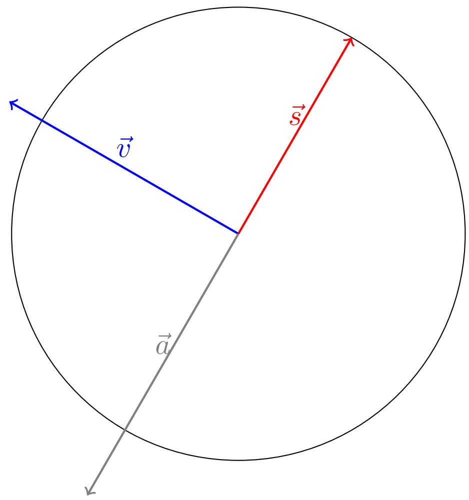

Figure 11: A phasor diagram containing the displacement, velocity and acceleration of a simple harmonic system at a particular instant. The displacement, velocity and acceleration phasors are in colours red, blue and grey respectively. Notice that the phasors are $\frac{\pi}{2}$ apart, and that the velocity and acceleration phasors have longer lengths due to the multiplication of the angular frequency defined by the individual harmonic system.

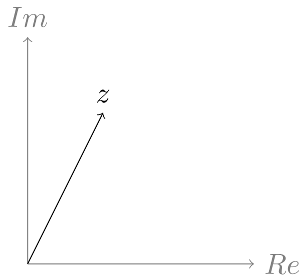

Figure 12: A complex number z on the *Argand plane*.

between different quantities of SHM.
To further simplify our analysis of SHM, I should draw your attention to the use of complex values: In Figure 12 the complex number $z = x + iy$ is shown in an *Argand diagram*. The projection on the real and imaginary axes are x and y respectively, and this is really useful in representing SHM - this is analogous to the phasors as stated previously! We can simply regard our SHM as an oscillation solely on the real or imaginary axes.
Let us first rewrite our complex number:

$$\begin{aligned} z &= x + iy \\ &= A\cos\theta + iA\sin\theta \\ &= A(\cos\theta + i\sin\theta) \end{aligned} \tag{71}$$

By using Euler's formula, we have:

$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta} \tag{72}$$

Replacing θ with $\omega t + \phi$, we have:

$$z = Ae^{i(\omega t+\phi)} \tag{73}$$

Notice that we have simplified our expression to a minimal by using complex numbers. This z number would spin around the origin with an initial phase angle ϕ just as a phasor, and to recover the original solutions, we only need to take the *real* (or imaginary) part of the solution:

$$Re\{z\} = Re\{Ae^{i(\omega t+\phi)}\} = A\cos(\omega t + \phi) \tag{74}$$

So why is the use of complex numbers so useful? We may notice its significance when we try to obtain the velocity and acceleration in the same way as before by differentiating the displacement:

$$z = Ae^{i(\omega t+\phi)} \tag{73}$$

$$\dot{z} = i\omega Ae^{i(\omega t+\phi)} \tag{75}$$

$$\ddot{z} = -\omega^2 Ae^{i(\omega t+\phi)} \tag{76}$$

These are exactly the same when you consider the real parts as the equations stated at the start of this section. By noticing the transformation of multiplying i to a complex number in an Argand digram (rotating it by $\frac{\pi}{2}$, we have hence obtained the same result from phasor analysis. This, in fact, is much simpler as we do not need to deal with the cumbersome trigonometric identities, especially when we consider SHM systems with multiple oscillations (combinations of SHM).
Notice how the complex solution satisfy the complex fundamental differential equation of SHM:

$$\ddot{z} = -\omega^2 z \tag{77}$$

Let us now turn to the analysis of energy using complex numbers. Potential energy and kinetic energy of the system can be obtained by:

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}k\ Re\{z\}^2 \tag{78}$$

$$KE = \frac{1}{2}k\dot{x}^2 = \frac{1}{2}k\ Im\{z\}^2 \tag{79}$$

By realising that $Re\{iz\} = -Im\{z\}$, we have recovered the original real expressions:

$$PE = \frac{1}{2}kA^2\cos^2(\omega t + \phi) \tag{49}$$

$$KE = \frac{1}{2}kA^2\sin^2(\omega t + \phi) \tag{50}$$

Hence,

$$\begin{aligned} E_{total} &= PE + KE \\ &= \frac{1}{2}k(Re\{z\}^2 + Im\{z\}^2) \\ &= \frac{1}{2}k|z|^2 \end{aligned} \tag{80}$$

where $\mid z \mid$ is the modulus of the complex number, geometrically, the radius of the vector of the complex number in the Argand diagram. This completely agrees with our previous results since:

$$\mid z \mid = A \tag{81}$$

where A is the amplitude of the oscillation.

Moving on

You are now equipped with the necessary knowledge to understand some harder physical concepts, such as damped and forced harmonic motion. To do that, however, you must first be comfortable in solving differential equations. Please look forward to a following-up document titled *ODEs and Oscillations* to be released next year. You are always welcome to ask me any questions at *lucasleung0149@gmail.com* or if you just want to help me complete my notes!

Best wishes,
Lucas Leung