Problem Book on Classical Mechanics

A Collection of QCPhO Example Classes Questions

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Contents

Preface

This book arose from a summer physics course in 2020 during the height of the Covid-19 pandemic. The aim of the course was to introduce classical mechanics to students who were interested and capable of studying lower-division university applied mathematics and physics. This book is a collection of the problems I have used in example classes held once a week to allow students to practice on questions related to topics they have just learnt in lessons.

Each chapter begins with a short summary of the theory used in solving the questions. Questions following this introduction develop the theory in more specific contexts. These questions are often stated in a more explicit form to appeal to students who are more familiar to the presentation styles typically used in schools. The solutions to the problems then follows, with a restatement of problems in a more concise form to mimic those seen in university-level texts and exams. Students will gain a better insight of the theory if they try solving the problems first without referring to the solutions; if they cannot solve it after numerous attempts the solutions can then be referred and students can then learn from the experience. Those who can solve the problems without referring to the solutions can also gain a better insight by reading the commentary in the solutions to the problems which often cover more general cases and arguments.

I am deeply indebted to my friends Hillman Lai, Ricky Li, Matthew Tong and Thomas Yuen for their kind suggestions and encouragement while writing the book. I must also thank the students who attended the course in pointing out the numerous mistakes I have made and for attempting the first few chapters of the books to allow a general discussion for responses to be made.

Lucas Leung
September 2020

Chapter 1

Elastic Collisions

One important type of questions that we did not cover in the test is related to collisions. These questions typically require you to evoke some sort of conservation laws and you have to be really clear about what physical situations we are talking about. In considering these problems we typically regard the system as *mechanically-isolated*, so there are no-external forces acting on the system. this gives the *Conservation of Momentum* as follows:

Law 1.1. *(Conservation of Momentum) Linear Momentum* $\mathbf{p}$ *is conserved if there are no external forces acting on the system such that* $\Sigma\mathbf{F_{ext}} = \mathbf{0}$.

Sometimes it might be explicitly stated that mechanical energy is conserved. Another way to say this is to say the collision is *elastic*. Make sure you remember these words before you continue.

1.1 Questions

Question 1: First we will consider two particles (with negligible dimensionality) **elastically** colliding in an isolated system. Let us suppose they have the same mass m for now. If the first particle has an initial velocity of u and collides with the second particle initially at rest head-on (Figure 1.1), find the final velocity of the two particles. Is your solution unique?

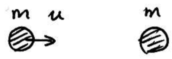


Figure 1.1: Two identical particle colliding.

Question 2: Now let us consider the two particles with different masses m_1 and m_2. They are aligned head-on (You may regard the problem as 1D collisions for now.) Suppose that the first particle has an initial speed of u while the second one is initially at rest (Figure 1.2).

(a) Choose a convenient frame to analyse the problem. Why is this frame important/convenient?

(b) Now obtain the velocity of the centre of mass in the laboratory frame. Transform the physical quantities in this frame.

(c) Analyse the conservation laws in this frame if the collision is **elastic**. You should obtain some important condition in this frame.

(d) Calculate the final velocities of the two particles in terms of m_1,m_2 and u.

At this stage it is useful to analyse the specific limits of the problem. Consider the following two limiting cases,

1. $m_1 \gg m_2$.
2. $m_1 \ll m_2$.

What do they mean physically? Do they match up intuitively? Why?

Note that the analysis above also applies when the initial speed of the particles are both not zero.

Figure 1.2: Two particle collision in 1D.

Question 3: Now consider two identical disks on a frictionless table (the collision problem becomes two-dimensional). If the collision is not head-on but elastic, show that the balls travel at orthogonal (perpendicular) directions after the collision (Figure 1.3).

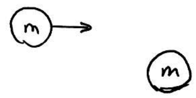


Figure 1.3: Two identical disks colliding in 2D plane.

Question 4: Let us analyse more on the situation I posed in Question 3. Suppose disks are dimensionless (so you cannot calculate the scattering angle explicitly) and have different masses. The first disk of mass m_1 has, again, an initial velocity u while the second one m_2 is originally at rest (Figure 1.4). The collision is **elastic**.

(a) Convert the system quantities into the centre-of-mass frame.

(b) What is the condition for the final velocities for the particles if the collision is elastic? Justify your answer.

(c) Drawing a vector diagram, show the relationship between the original velocity vector in lab frame, the original velocity vector in COM frame and the velocity vector of COM for both particles. Repeat the same for the final velocity, supposing that the final velocity vectors can point to any direction of your choice.

(d) Hence deduce the largest recoil angle of the first particle (labelled m_1 in Figure 1.4) in LAB frame from the horizontal in terms of the given quantities (ψ is the recoil angle in the COM frame/ ZMF (known as the parameter angle), see Figure 1.4).

I will discuss a different method to analyse this problem in the class.

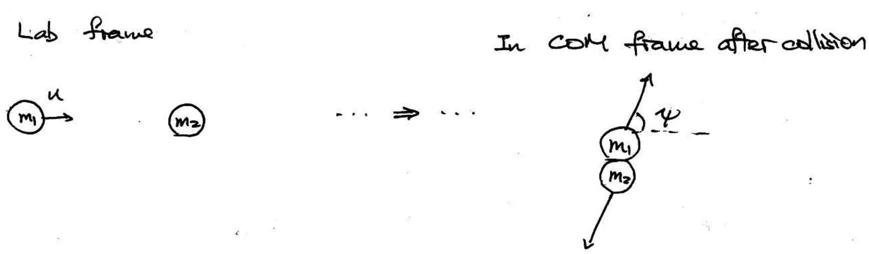


Figure 1.4: Some sketches showing the set-up in Question 4. Note that the angle ψ is the scattering angle in the COM frame/ ZMF.

1.2 Solutions and Commentary

Note that the questions in these sections are tidier - they give you less hints on what you should do. This is just my way of analysing questions so the answers for individual parts may not be explicitly shown - you are left to figure that out yourself.

Question 1: Consider two particles (with negligible dimensionality) of the same mass m **elastically** colliding in an isolated system. If the first particle has an initial velocity of u and collides with the second particle initially at rest head-on (Figure 1.1), find the final velocity of the two particles.
Solution: This is a specific case of the following question. See the solution below. Conserving energy and momentum should give you a solution where $v_1 = 0$ and $v_2 = u$, v_1 and v_2 being the final velocities of the two particles respectively. The thing I want you to notice is that since we are solving a quadratic equation, this is not mathematically the unique solution as $v_1 = u$ and $v_2 = 0$ also satisfies the conservation laws. You can typically physically eliminate that (since that would mean that the particles essentially pass through each other[1]) but you do have to explicitly state it out to show that you have considered that. This comes to full effect in the next question when you consider different cases in the ZMF (Zero-momentum Frame[2]).

Question 2: Consider the one-dimensional (head-on) collision of two particles with different masses m_1 and m_2. Suppose that the first particle has an initial speed of u while the second one is initially at rest (Figure 1.2). Calculate the final velocities of the particles and analyse different limits to check your answer.
Solution: Let us analyse in the frame of centre-of-mass. This is typically known as the zero-momentum frame (ZMF) since by construction the total momentum in the frame is zero:

Proposition 1.1. *The total momentum for a system of particles in the ZMF is* $\mathbf{0}$.

Proof. The position of the centre of mass is defined as

$$\mathbf{x_{COM}} = \frac{\Sigma_{i=1}^{N} m_i \mathbf{x_i}}{\Sigma_{i=1}^{N} m_i} \tag{1.1}$$

So the velocity of the COM is:

$$\dot{\mathbf{x}}_{\mathbf{COM}} = \frac{\Sigma_{i=1}^{N} m_i \dot{\mathbf{x}}_{\mathbf{i}}}{\Sigma_{i=1}^{N} m_i} \tag{1.2}$$

where $\dot{\mathbf{x}} = \dot{x}_i \mathbf{e_i}$ is the 3D velocity[3]. Now in ZMF $\dot{\mathbf{x}} = \mathbf{0}$, so $\Sigma_{i=1}^{N} m_i \dot{\mathbf{x}}_{\mathbf{i}} = \mathbf{0}$. □

[1] Sometimes this might occur - in Quantum Mechanics, not here though.

[2] Physicists like to use this word since that explicitly states the importance of the frame. It is synonymous to the Centre-of-mass Frame.

[3] Einstein Summation Convention used.

The COM velocity is

$$v_{COM} = \frac{m_1 u}{m_1 + m_2} \tag{1.3}$$

so we have

$$\begin{cases} u_1^{'} = \frac{m_2 u}{m_1+m_2} \\ u_2^{'} = -\frac{m_1 u}{m_1+m_2} \end{cases} \tag{1.4}$$

where the prime quantities are the respective quantities in the ZMF. Now we evoke the conservation laws. Conservation of energy and momentum gives:

$$\begin{cases} \frac{1}{2}m_1 u_1^{'2} + \frac{1}{2}m_2 u_2^{'2} = \frac{1}{2}m_1 v_1^{'2} + \frac{1}{2}m_2 v_2^{'2} \\ m_1 u_1^{'} + m_2 u_2^{'} = m_1 v_1^{'} + m_2 v_2^{'} = 0 \end{cases} \tag{1.5}$$

The second equation gives $u_2^{'} = \frac{m_1}{m_2} u_1^{'}$, etc. So simplifying gives

$$u_1^{'2} = v_1^{'2} \tag{1.6}$$

which can be written as

$$|u_1^{'}| = |v_1^{'}| \tag{1.7}$$

and similarly

$$|u_2^{'}| = |v_2^{'}|$$

. The speed of the particles in ZMF is conserved. Now we evoke the argument we used in the previous question - if the signs of the velocities are conserved then the particles would simplify"pass through" one another. So we set $u_i^{'} = -v_i^{'}$, where $i = 1, 2$. Then

$$\begin{cases} v_1^{'} = -\frac{m_2 u}{m_1+m_2} \\ v_2^{'} = \frac{m_1 u}{m_1+m_2} \end{cases} \tag{1.8}$$

Transforming back to lab frame gives

$$\begin{cases} v_1 = \frac{m_1-m_2}{m_1+m_2} u \\ v_2 = \frac{2m_1}{m_1+m_2} u \end{cases} \tag{1.9}$$

Now we check the limits. When $m_1 \ll m_2$ we have

$$\begin{cases} v_1 = -u \\ v_2 = 0 \end{cases} \tag{1.10}$$

which makes sense if you consider a particle elastically hitting the wall[4]. When $m_1 \gg m_2$ we have

$$\begin{cases} v_1 = u \\ v_2 = 2u \end{cases} \tag{1.11}$$

[4]Something to think about when you consider ideal gases.

The first velocity makes sense - it is like there is no interference to the motion of the first particle to the second particle due to the mass difference. The second result however is quite surprising - it is not infinite[5]!

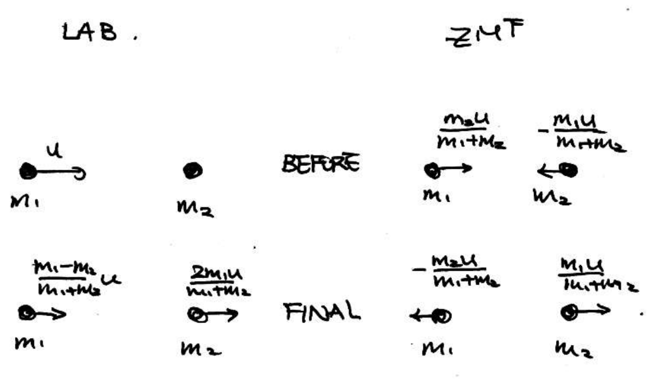


Figure 1.5: Sketches for the solution for Question 2.

Question 3: Consider two identical disks on a frictionless table (the collision problem becomes two-dimensional). If the collision is not head-on but elastic, show that the balls travel at orthogonal (perpendicular) directions after the collision (Figure 1.3).
Solution: The trick to use here is to draw a vector diagram. To conserve momentum, we need $\mathbf{v_1} + \mathbf{v_2} = \mathbf{u}$.

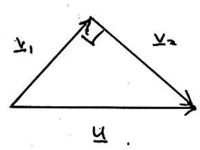


Figure 1.6: Vector diagram for 2D elastic collision.

Now conserving energy requires $u^2 = v_1^2 + v_2^2$. Reverse of Pythagoras' Theorem gives the $\frac{\pi}{2}$ argument.

[5]Actually, if you sit down and think about it - it physically makes sense. Just transform the problem into the COM frame and you will obtain something similar to a moving wall colliding with a stationary particle. In wall's frame, the particle has a velocity $-u$ so it rebounces, in this frame, with velocity u. Transforming to lab frame gives $2u$.

Question 4: Consider a 2D elastic collision of dimensionless particles. The first particle of mass m_1 has, again, an initial velocity u while the second one m_2 is originally at rest (Figure 1.4). Find the maximum recoil angle θ_1 of the first mass after the collision. *Find also the relationship between the scattering angle ϕ and the parameter angle ψ.*

Solution: One person asked why I want to particles to be dimensionless. This is to allow the parameter angle ψ become a free variable so you cannot guess that from the geometry[6]. From Question 2 we have that

$$|u_i^{'}| = |v_i^{'}|$$

for $i = 1, 2$ in the ZMF. Draw a vector diagram for the final velocities as follows.

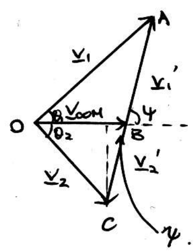


Figure 1.7: Vector diagram for dimensionless 2D elastic collision.

To maximise θ_1, the recoil angle for the first particle, we must consider different cases. When $|v_i^{'}| > |v_{COM}^{'}|$, the maximum recoil angle is obvious - it is backwards and $\Rightarrow \theta_1 = \pi$. So we consider the case $|v_i^{'}| < |v_{COM}^{'}|$. The maximum recoil angle is reached when vector $\mathbf{v_1}$ is orthogonal to vector $\mathbf{v_1^{'}}$

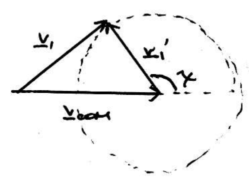


Figure 1.8: Condition for maximum θ_1 in the case of $|v_i^{'}| < |v_{COM}^{'}|$.

so

$$\theta_1 = \sin^{-1}\left(\frac{v_1^{'}}{v_{COM}}\right) = \sin^{-1}\left(\frac{m_2}{m_1}\right) \tag{1.12}$$

[6] This is actually an interesting exercise. Set the disks to have a radius R and suppose originally their centres are separated by a distance $d < 2R$. Calculate the parameter angle ψ by using law of reflection.

Now let us find the relationship between the scattering angle ϕ and the parameter angle ψ. From Figure 1.7 we have

$$\tan\theta_1 = \frac{v_1 \sin\psi}{v_1 \cos\psi + v_{COM}} \tag{1.13}$$

$$\tan\theta_2 = \frac{v_2 \sin\psi}{-v_2 \cos\psi + v_{COM}} \tag{1.14}$$

We can further simplify this. Notice that $|v_2'| = |v_{COM}|$. So $\triangle OBC$ is isosceles and

$$\theta_2 = \frac{1}{2}(\pi - \psi) \tag{1.15}$$

Therefore, it follows that

$$\tan\theta_2 = \cot\frac{\psi}{2} \tag{1.16}$$

Substitution for the velocities give

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \frac{m_1}{m_2}} \tag{1.17}$$

So the scattering angle is

$$\tan\phi = \tan(\theta_1 + \theta_2) \tag{1.18}$$

Here you can use the *t-substitution* $t = \tan\frac{\psi}{2}$:

$$\begin{cases} \sin\psi = 2\sin\frac{\psi}{2}\cos\frac{\psi}{2} = \frac{2t}{1+t^2} \\ \cos\psi = 2\cos^2\frac{\psi}{2} - 1 = \frac{1-t^2}{1+t^2} \\ \tan\psi = \frac{2\tan\frac{\psi}{2}}{1-\tan^2\frac{\psi}{2}} = \frac{2t}{1-t^2} \end{cases} \tag{1.19}$$

Then we have

$$\tan\phi = \frac{m_1 + m_2}{m_1 - m_2}\cot\frac{\psi}{2} \tag{1.20}$$

You can check that the maximum scattering angle is π, neglecting any relativistic effects. This matches up with our expectations.

1.3 Take-home skills

The skills that you should now know is:

1. Checking Conditions.
2. Using ZMF in collision problems.
3. Understanding the importance of vector diagrams in analysing problems.

1.4 Response Comments

General Comments

1. Q1: In general the discussion of the uniqueness condition is not clear. My fault on setting the question in an ambiguous way.

2. Q2: Most people successfully obtained the answer. Surprisingly, some thought the COM frame allows the "exchange" of velocities which is wrong - if you evoke conservation laws in this frame you will check that this is incorrect. For that some marks are deducted (although they all got the correct answer in the end...?) Intuitive descriptions are expected in the limiting case part. The best answers sketched out why they think the limit cases made sense.

3. Q3: Nobody did the question correctly. Some got really close, and two evoked some argument from the conditions in Q2 in the ZMF although they are now working in the Lab frame.

4. Q4: Generally poor attempts. Some did't even try to copy their correct answers from Q2 (Two marks missed there!) and did not attempt the question at all. Others drew ambiguous vector diagrams that are incorrect. One got very close with the correct vector diagram but then digressed into something else.

5. In general this exercise is attempted to my expected standards. An approximate bimodal distribution can be deduced - which can be attributed to the HKDSE-level training that S4 students experience. For S3 students the exercise is a good challenge but hopefully they will learn more from that.

(Updated by Lucas on 25$^{\text{th}}$ July, 2020)

Chapter 2

Ladders, Blocks and Stability

In high-school physics you should have learnt about the conditions for mechanical equilibrium. Here is an elementary definition.

Definition 2.1. *An object in **mechanical equilibrium** satisfies:*

$$\Sigma \mathbf{F_{ext}} = \mathbf{0} \tag{2.1}$$

$$\Sigma \mathbf{G_{ext}} = \mathbf{0} \tag{2.2}$$

In the first Newtonian Mechanics lecture I gave, I also described how mechanical stability can be analysed in terms of the curvature of the *potential energy function.* You should recall the following:

Definition 2.2. *Suppose an object has a potential energy function $V(x)$ in 1D. If $\frac{d^2V}{dx^2}(x_0) > 0$, the object is said to be in **stable equilibrium** at $x = x_0$. If $\frac{d^2V}{dx^2}(x_0) < 0$, the object is said to be in **unstable equilibrium** at $x = x_0$.*

In this example class we will primarily investigate how mechanical stability can be analysed without referring to the potential energy function. This relies on drawing accurate force diagrams. We will primarily analyse two kinds of systems seen everywhere in physics - ladders and blocks.

2.1 Questions

Question 5: We will start with an extremely easy question. Consider a block of mass m and dimensions $a \times a \times b$ $(b > a)$ put on a rough slope as shown in Figure 2.1.

(a) Draw a representative free body diagram for the block.

(b) What is the maximum angle that the block can be in stable equilibrium (You can move the block)? You may assume that the static friction coefficient μ_s is infinitely large to neglect any sliding motion of the block.

(c) Thomas and Lucas are good friends. Thomas is 6 ft tall and Lucas is 5'6". You may assume here they can be modelled as blocks (with the same base area to first order correction). Say they are climbing a slope. Which of them is more likely to fall if they suddenly stand still (their whole body is orthogonal to the slope)? Why?

(d) Now assume the same block placed on the ground with the square-side touching the ground (Figure 2.2). Find the conditions such that the block just slides and just topples respectively if a force F is given at the tip. The static friction coefficient between the block and the flat surface is μ_0.

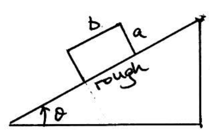


Figure 2.1: A 2D representation of a block put on a rough incline. The block is not fixed and you can move the block.

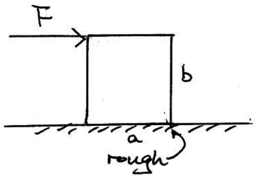


Figure 2.2: The same block is put on a rough surface with a force F acted on it.

Question 6: Consider a cuboid put on a hemisphere of radius R (Figure 2.3). We wish to find the conditions for the dimensions of the block for which the block is in stable equilibrium.

(a) Suppose the block is perturbed by a small angle θ. Draw a force diagram for the block at this angle. [*Hint: notice that the contact point of the two objects is no longer the centre of the side!*]

(b) Using your knowledge from elementary physics analysis, draw a force diagram with a representative dimension of the block. Then obtain the condition for the block to not topple over.[*Hint: Something to do with centre of mass perhaps?*]

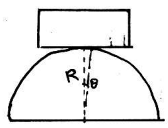


Figure 2.3: A cuboid put on a hemisphere of radius R. The perturbation angle is shown as θ.

Question 7: N identical blocks of length L, width W and height H. They are stacked together at the edge of the table such that they are just about to topple (Figure 2.4). The extension length l_0 is defined as the length from the edge of the table to the outer edge of the block furthest away from the table. Find l_0 when $N = 4$. Describe the behaviour when $N \to \infty$.

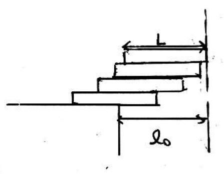


Figure 2.4: N blocks are stacked on top of each other such that the topmost block is placed furthest away from the table. Here $N = 4$.

Question 8: A ladder of uniform density (mass per unit length) ρ and length L is leaning against the wall.

(a) Suppose the surface between the ladder and the wall is smooth while that between the ladder and the ground is rough. If the static friction coefficient between the ladder and the ground is μ, find the smallest angle ϕ that the ladder can make with the ground and not slip (Figure 2.5).

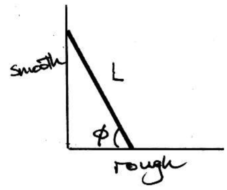


Figure 2.5: Ladder leaning against the wall. The surface between the ground and the ladder is rough while that between the ladder and the wall is smooth.

(b) Suppose now the surface between the ladder and the wall is rough while that between the ladder and the ground is smooth. Suppose the ladder is just about the slide against the wall. If the static friction coefficient between the ladder and the wall is μ, and a string attaches from the centre of mass, find the tension in the string (Figure 2.6).

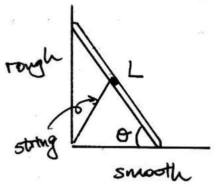


Figure 2.6: Ladder leaning against the wall. The surface between the ground and the ladder is smooth while that between the ladder and the wall is rough. A string connects from the centre of mass to the corner of the wall and ground.

(c) Suppose now all surfaces are rough. Try drawing a force diagram and describe why force analysis cannot be performed here. [*Hint: Count the degrees of freedom in the question.*]

Question 9: A uniform ladder of mass m and length L is leaning against the wall. All surfaces are smooth. Denote the point on the ladder in contact with the wall as P. The starting angle that the ladder makes with the **wall** is θ_0 (Figure 2.7 and 2.8). If you are not familiar with rigid body dynamics, you are allowed to treat the ladder as a massless rod connected to two small spheres at its ends with masses $\frac{m}{2}$. You are allowed to choose one of the following methods.

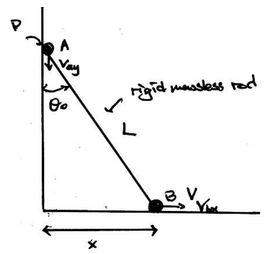


Figure 2.7: A model of a ladder leaning against the wall. All surfaces are smooth.

(I) (*HKPhO-level*) This method does not require the use of calculus or rigid body-dynamics (Figure 2.7).

(a) Find the kinetic energy and potential energy of ladder. Using conservation of energy, find an expression that links the velocities of the end masses v_{bx} (bottom, horizontal), v_{ay} (upper, vertical) and the angle θ.

(b) The rod is a rigid body. What condition must the rod satisfy at all times?

(c) Using the relation in (b), hence obtain an equation linking the horizontal velocity of the bottom particle v_{bx} and the angle θ.

(d) What is the condition for point P (of the rod) to detach from the wall?

(e) Using the fact that the maximum of the function $y(x) = Ax^2(a - x)$ is at $x = \frac{2}{3}a$, where A and a are constants, find the respective angle for which the ladder detaches from the wall.

(II) (*semi-HKPhO level*) This method can be done with a bit of calculus and your HKPhO knowledge (Figure 2.7).

(a) Find the kinetic energy and potential energy of ladder. Using conservation of energy, find an expression that links the velocities of the end masses v_{bx} (bottom, horizontal), v_{ay} (upper, vertical) and the angle θ.

(b) The rod is a rigid body. What condition must the rod satisfy at all times?

(c) Using the relation in (b), hence obtain an equation linking the horizontal velocity of the bottom particle v_{bx} and the angle θ.

(d) What is the condition for point P (of the rod) to detach from the wall?

(e) Obtain a relation that links the x-position of the bottom particle and the angle θ. Hence express the equation and condition found in part (c) and (d) respectively as a functional[1] of θ and its derivatives.

(f) Differentiating the equation in (c) (as a functional of θ) with respect to t, find the equation of motion of the system. Using this equation, and the two from part (e), find the angle for which the ladder detaches from the wall.

[1]This means a function of functions.

(III) (*Standard Method*) This requires rigid body dynamics and calculus. Don't do it unless you are hundred-percent sure what you are doing (although I would have chosen to do this when I was your age).

(a) Find the moment of inertia of the ladder about its centre of mass.

(b) Find the kinetic energy and potential energy expressions of the rod when the angle between the ladder and the **wall** is θ. You can take any reference points you like for the potential energy expression but make sure you state it.

(c) Find the relationships between the centre of mass coordinates x and y and the angle θ.

(d) Using the fact that energy is conserved, obtain a relation linking θ with its derivative with time.

(e) Either perform a force analysis[2] of the problem or continue using the conservation of energy you are using. Obtain an equation of motion of the system.

(f) Find the physical condition for the ladder to detach from the wall. Hence find the angle for this to occur.

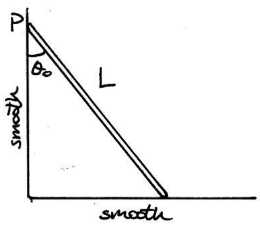


Figure 2.8: Ladder leaning against the wall. All surfaces are smooth.

[2]You must state why your equations are valid if you obtain them in a non-inertial frame.

2.2 Solutions and Commentary

Question 5: Consider a block of mass m and dimensions $a \times a \times b$ ($b > a$) put on a rough slope as shown in Figure 2.1. What is the maximum angle that the block can be in stable equilibrium? If the same block is then placed on the ground with the square-side touching the ground (Figure 2.2), find the conditions such that the block just slides and just topples respectively if a force F is given at the tip. The static friction coefficient between the block and the flat surface is μ_0.

Solution: This is a rudimentary problem on statics and the equilibrium condition. The main thing I would like you to know is that in analysing these stability questions, the *normal reaction* on your body need not be situated at the middle of the contact surface, but can be theoretically moved to any contact point depending on the situation we are in. We start by drawing free body diagrams of the box.

(a) Short side as base. (b) Long side as base.

Figure 2.9: Free-body diagram of the box at the critical point with two configurations.

Here f is the static frictional force experienced by the block. First consider Figure 2.9a. By considering the equilibrium conditions, we can deduce that the block is just about to topple when the centre of mass is just above the lower end of the block (point P in Figure 2.10). This therefore gives the condition:

$$\tan\theta = \frac{a}{b} \tag{2.3}$$

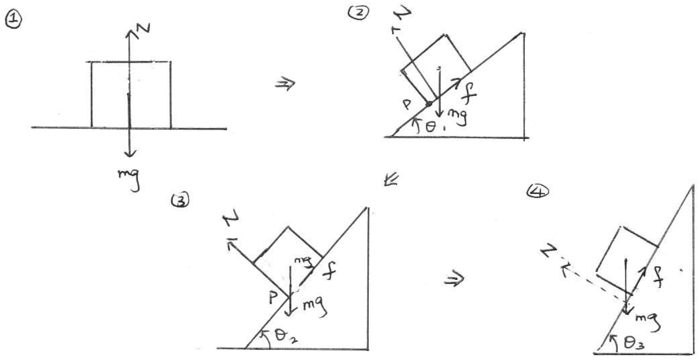


Figure 2.10: Free-body diagrams of the blocks as θ is increased. Let us consider the balance of moments at the lowest corner of the block (point P). ①: When θ is zero, the normal force is antiparallel to the weight and it passes through the centre of mass of the block. ②: As we increase θ, the weight gives a clockwise moment at point P. Since the frictional force does not contribute to the external torque at P, for the block to be in equilibrium, we require the normal force to provide an anticlockwise moment at P. ③: The critical condition is reached when the centre of mass is directly on top of point P and the normal reaction reaches the edge. ④: If θ is further increased, we would require the normal reaction to be outside the box to balance the anticlockwise moment caused by the weight. This is not physically possible, so the block topples over.

However, we have not considered all cases. If we consider the second case illustrated in Figure 2.9b, we will find that the condition for the box to just not topple as

$$\tan\theta = \frac{b}{a} \tag{2.4}$$

Comparing Equations 2.3 and 2.4 with $b > a$ gives Equation 2.4 as the final condition. It is therefore evident that a shorter person is better at balancing[3].

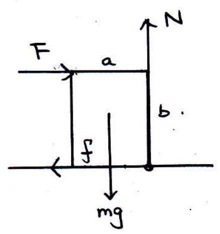


Figure 2.11: Force diagram of the block when acted by a force F.

Now consider the block put on a rough flat surface. For the block to just about to slide, we take the largest value possible for the static friction. So, balancing

[3] Apparently the only good thing for short people like me.

forces give

$$F_{slide} = \mu mg \tag{2.5}$$

When the block is about the topple, the normal reaction is displaced so it is at the far corner of the lower surface of the block. Balancing moments at that point gives

$$F_{topple} = \frac{mga}{2b} \tag{2.6}$$

Question 6: Consider a cuboid put on a hemisphere of radius R (Figure 2.3). Find the conditions for the dimensions of the block for which the block is in stable equilibrium.

Solution: Suppose a cuboid of height h is perturbed from equilibrium by a small angle θ. Assume that the block does not slip during this process. Then drawing a diagram at this perturbed position:

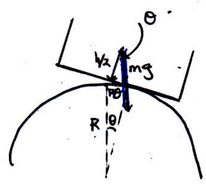


Figure 2.12: Force diagram for the block of height h perturbed by a small angle θ. The blue arrow indicates the weight vector of the block.

Making a similar analysis from Figure 2.10, we can deduce from Figure 2.12 that the critical condition occurs when the centre of mass is just above the contact point. From Figure 2.12 we have for stable equilibrium:

$$\tan\theta < \frac{2R\theta}{h} \tag{2.7}$$

Using small angle approximation $\theta \approx \tan\theta$, we have

$$h < 2R \tag{2.8}$$

This is the condition for stable equilibrium.

Potential energy function Another way of analysing the problem is to write out the potential energy function.

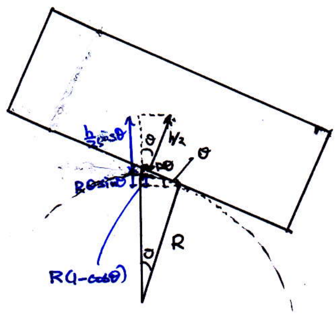


Figure 2.13: Detailed lengths in the block-hemisphere problem.

From Figure 2.13, we can write:

$$V = \frac{h}{2}\cos\theta + R\theta\sin\theta - R(1-\cos\theta) \tag{2.9}$$

where V is normalised by mg (i.e. $U = mgV$). Now we use the small angle approximation,

$$\begin{cases} \sin\theta = \theta + o(\theta^2) \\ \cos\theta = 1 - \frac{1}{2}\theta^2 + o(\theta^2) \end{cases} \tag{2.10}$$

Notice that we have truncated the series at second-order terms as we would like to differentiate the function by two times. This gives

$$V = \frac{h}{2}(1 - \frac{\theta^2}{2}) + R\frac{\theta^2}{2} \tag{2.11}$$

According to Definition 2.2, we differentiate V with respect to θ to give:

$$\frac{dV}{d\theta} = -\frac{h\theta}{2} + R\theta = 0 \tag{2.12}$$

so equilibrium occurs at $\theta = 0$. Now differentiate again with respect to θ gives:

$$\frac{d^2V}{d\theta^2} = -\frac{h}{2} + R \tag{2.13}$$

For stable equilibrium we want $\frac{d^2V}{d\theta^2} > 0$ so this evaluates to

$$h < 2R \tag{2.8}$$

as before.

Question 7: N identical blocks of length L, width W and height H. They are stacked together at the edge of the table such that they are just about to topple (Figure 2.4). The extension length l_0 is defined as the length from the edge of the table to the outer edge of the block furthest away from the table. Find l_0 when $N = 4$. Describe the behaviour when $N \to \infty$.
Solution: Begin our analysis from the topmost block. There are only two forces acting on the system, the weight and the normal reaction. To satisfy the conditions in Equation 2.1, we require the two to be antiparallel and pass through the centre of mass of the block (Figure 2.14). If we now consider the next block, for the block to be positioned the furthest out, the reaction normal force N_1 must then act on the edge of the next block. By Equation 2.1, you can compute the location of N_2[4].

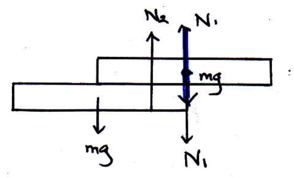


Figure 2.14: Diagram for the first two blocks. Blue arrows are forces acting on the first block, black arrows are for that acting on the second block. Note that this is not a valid force diagram.

We analyse the general condition for the system. Consider the (N+1)$^{\text{th}}$ block. We can draw the force diagram as follows:

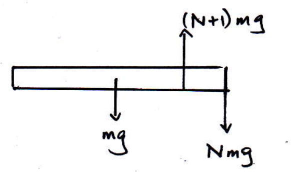


Figure 2.15: Force diagram of the (N+1)$^{\text{th}}$ block.

Suppose the reaction force pointing upward is x from the centre of mass of this block. Equation 2.1 gives

$$(N+1)mgx = Nmg\frac{L}{2} \tag{2.14}$$

Solving for x gives:

$$x = \frac{N}{N+1}\frac{L}{2} \tag{2.15}$$

[4]Another way of seeing this is that we require any upper subsystem (that includes the entire collection of blocks above, say the N$^{\text{th}}$ block from the top) to have a centre of mass lying at most at the edge of the immediate block below it.

The distance l that N$^{\text{th}}$ block can be displaced is:

$$l = \frac{L}{2} - x = \frac{L}{2(N+1)} \tag{2.16}$$

So for the N blocks, the extension length l_0 is:

$$l_0 = \frac{L}{2} \sum_{k=1}^{N} \frac{1}{k} \tag{2.17}$$

For $N = 4$, this evaluates to $l_{0,N=4} = \frac{25L}{24}$. When $N \to \infty$, Equation 2.17 becomes:

$$l_0 = \frac{L}{2} \sum_{k=1}^{\infty} \frac{1}{k} \tag{2.18}$$

The right-hand side contains a harmonic series and has no finite limit. The blocks can be extended to infinity.

Question 8: A ladder of uniform density (mass per unit length) ρ and length L is leaning against the wall.

(a) Suppose the surface between the ladder and the wall is smooth while that between the ladder and the ground is rough. If the static friction coefficient between the ladder and the ground is μ, find the smallest angle ϕ that the ladder can make with the ground and not slip (Figure 2.5).

(b) Suppose now the surface between the ladder and the wall is rough while that between the ladder and the ground is smooth. Suppose the ladder is just about the slide against the wall. If the static friction coefficient between the ladder and the wall is μ, and a string attaches from the centre of mass, find the tension in the string (Figure 2.6).

(c) Suppose now all surfaces are rough. Describe why force analysis cannot be performed here.

Solution: First consider the case where only the floor is rough. The force diagram of the ladder is:

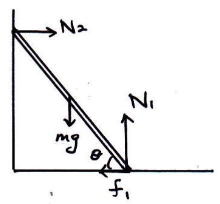


Figure 2.16: Force diagram of the ladder when only the floor is rough.

Equation 2.1 gives

$$\begin{cases} N_1 = mg \\ f_1 = N_2 \\ N_2 L \sin\theta = mg\frac{L}{2}\cos\theta \end{cases} \tag{2.19}$$

taking the reference point for torques at the point of contact of the ladder and the ground. Simplifying and using the condition $f_1 \leq \mu N_1$ gives

$$\tan\theta \geq \frac{1}{2\mu} \tag{2.20}$$

The required smallest angle can hence be found.

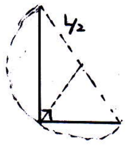


Figure 2.17: By considering the ladder as the diameter of the circle, circle geometry gives that the string has a length equal to the radius of this circle (the reference is converse of $\angle$ in semicircle).

Now consider the case where only the wall is rough. Circle geometry gives that the length of the string be $\frac{L}{2}$ (see Figure 2.17). The force diagram of the ladder is

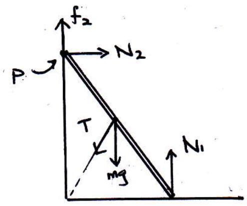


Figure 2.18: Force diagram of the ladder when only the wall is rough.

Equation 2.1 gives

$$\begin{cases} T\cos\theta = N_2 \\ N_1 + f_2 = mg + T\sin\theta \\ mg\frac{L}{2}\cos\theta + T\frac{L}{2}\sin 2\theta = N_1 L\cos\theta \end{cases} \tag{2.21}$$

taking the reference point for torques at the point of contact of the ladder and the wall (point P in Figure 2.18). Using $f_2 = \mu N_2$ (just about to slip) and simplifying gives

$$T = \frac{mg}{2\mu\cos\theta} \tag{2.22}$$

For the last case where all surfaces are rough, the force diagram is:

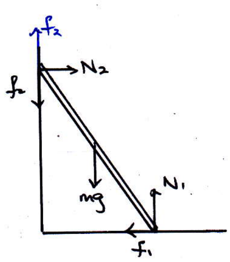


Figure 2.19: Force diagram of the ladder when all surfaces are rough.

Notice that typically we have the friction coefficients as constants. So the free variables of the system are N_1, N_2. However, the problem arises when we set our conditions. If I say the ladder is about to slip, which contact point does it refer to? This introduces a third degree of freedom. The fourth degree of freedom comes from the uncertainty in the direction of f_2. So in general, we have at least four degrees of freedom with three equations to solve - a family of solution is obtained rather than a unique one.

Question 9: A uniform ladder of mass m and length L is leaning against the wall. All surfaces are smooth. Denote the point on the ladder in contact with the wall as P. The starting angle that the ladder makes with the **wall** is θ_0 (Figure 2.20). Find the angle when the ladder detaches from the wall.

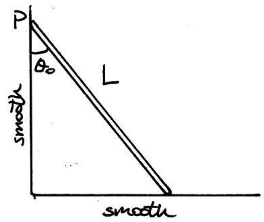


Figure 2.20: Ladder leaning against the wall. All surfaces are smooth.

Solution: Honestly I did not know how hard this question is until I tried it myself! This is actually a very standard question taught in universities but I have neglected the fact that you need to fully understand rigid body dynamics to do the question. However, with suitable hints, it can be reduced to a HKPhO-level question. I have presented 3 ways of tackling the problem and I will go through all of them here.

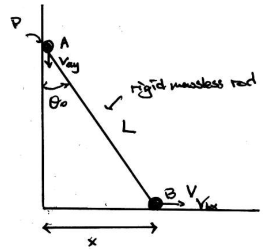


Figure 2.21: A model of a ladder leaning against the wall. All surfaces are smooth.

Method I We begin with standard HKPhO analysis (refer to Figure 2.21, we treat the ladder as a rigid massless rod connected to two balls of mass $\frac{m}{2}$). It is obvious that here you should use conservation of energy[5]. So using conservation

[5]I say obvious - it is because years of intuition make me think so. But you will soon realise there are not really other choices.

of energy:

$$\frac{1}{2}mgl(\cos\theta_0 - \cos\theta) = \frac{1}{4}m(v_{bx}^2 + v_{ay}^2) \tag{2.23}$$

which reduces to

$$2gl(\cos\theta_0 - \cos\theta) = v_{bx}^2 + v_{ay}^2 \tag{2.24}$$

Note that any horizontal motion of ball A can be ignored since the ball remains falling vertically until it is detached from the wall. Since the rod is a rigid body, every point of the rod (including the balls) must have the same motion (except motion about the centre-of-mass, for example circular motion about the centre of mass). This means that the velocity along the rod must be the same, and we have the condition:

$$v_{ay}\cos\theta = v_{bx}\sin\theta \tag{2.25}$$

which reduces to

$$v_{ay} = v_{bx}\tan\theta \tag{2.26}$$

When ball A detaches from the wall, the normal reaction between ball A and the wall disappears. This is because the outward force provided by the rod disappears as the rod is no longer compressed. As the rod starts to contract, the horizontal acceleration at that point is zero. In other words, v_{bx}^2 reaches it maximum at this moment. Combining Equation 2.24 and 2.26 gives ($\tan^2\theta + 1 = \sec^2\theta$)

$$v_{bx}^2 = 2gl\cos^2\theta(\cos\theta_0 - \cos\theta) \tag{2.27}$$

All we have to do is minimise Equation 2.27 at this point. Luckily we know that the maxima of function $y(x) = 2glx^2(a - x)$ is at $x = \frac{2}{3}a$. (To check, differentiate $y(x)$ and set it to zero.

$$\frac{dy}{dx} = 4glx(a - x) - 2glx^2 = 0$$

You can check that this is a maxima by invoking the second derivative rule.) So we can immediately put down the condition for detachment.

$$\cos\theta = \frac{2}{3}\cos\theta_0 \tag{2.28}$$

The angle of detachment can hence be found.

Method II Now let us use a bit of calculus. The first steps are identical to the first method. First obtain an equation from conservation of energy:

$$\frac{1}{2}mgl(\cos\theta_0 - \cos\theta) = \frac{1}{4}m(v_{bx}^2 + v_{ay}^2) \tag{2.23}$$

which reduces to

$$2gl(\cos\theta_0 - \cos\theta) = v_{bx}^2 + v_{ay}^2 \tag{2.24}$$

At this point, again use the fact that the rod is a rigid body. Then it follows that the velocities along the rod are the same:

$$v_{ay}\cos\theta = v_{bx}\sin\theta \tag{2.25}$$

which reduces to

$$v_{ay} = v_{bx}\tan\theta \tag{2.26}$$

Combining Equation 2.24 and 2.26 gives ($\tan^2\theta + 1 = \sec^2\theta$)

$$v_{bx}^2 = 2gl\cos^2\theta(\cos\theta_0 - \cos\theta) \tag{2.27}$$

At this point the analysis departs from the intuitive argument in Method I. Denote the position of ball B by x. Simple trigonometry gives

$$x = l\sin\theta \tag{2.29}$$

Consider the entire ladder as a system. The only force causing horizontal acceleration is the reaction force on ball A acted by the wall (remember the pulling/stretching force of the rod is internal to the system!). When the ladder detaches from the wall, this reaction force is zero so the horizontal acceleration of the system is zero. Since the centre of mass and ball B is fixed on the same rod (same horizontal acceleration), we have the condition

$$\ddot{x}_{COM} = \ddot{x} = 0 \tag{2.30}$$

Differentiating Equation 2.29 gives

$$\dot{x} = l\cos\theta\dot{\theta} \tag{2.31}$$

$$\ddot{x} = -l\sin\theta\dot{\theta}^2 + l\cos\theta\ddot{\theta} = 0 \tag{2.32}$$

so we have the condition for detachment as:

$$\ddot{x} = \tan\theta\dot{\theta}^2 \tag{2.33}$$

Using Equation 2.31 on 2.27, where $v_{bx} = \dot{x}$, we have

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta_0 - \cos\theta) \tag{2.34}$$

Differentiating Equation 2.34 by time gives:

$$\ddot{\theta} = \frac{g}{l}\sin\theta \tag{2.35}$$

Now we have θ, $\dot{\theta}$ and $\ddot{\theta}$ as variables. Using Equations 2.33, 2.34 and 2.35 reduces to the angular condition for detachment

$$\cos\theta = \frac{2}{3}\cos\theta_0 \tag{2.28}$$

as found in Method I.

Method III - Energy Method This is the most standard method to solve the problem. The moment of inertia of the rod is $I = \frac{1}{12}mL^2$. To obtain this, suppose the density of the rod is $\rho = \frac{m}{L}$. Then, calculation of moment of inertia proceeds as follows:

$$I = \int r^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho x^2 dx = \frac{1}{12}mL^2 \tag{2.36}$$

The coordinates of the centre of mass is at $(x, y) = (\frac{L}{2}\sin\theta, \frac{L}{2}\cos\theta)$. Using conservation of energy, we have:

$$\frac{1}{2}mgl(\cos\theta_0 - \cos\theta) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \tag{2.37}$$

Simplifying and using that $(\dot{x}, \dot{y}) = (\frac{L}{2}\dot{\theta}\cos\theta, -\frac{L}{2}\dot{\theta}\sin\theta)$ gives:

$$\dot{\theta}^2 = \frac{3g}{L}(\cos\theta_0 - \cos\theta) \tag{2.38}$$

To obtain the equation of motion of the system, note that conservation of energy implies that $\dot{E} = 0$. This is known as *the energy method*. Differentiating Equation 2.38 gives:

$$\ddot{\theta} = \frac{3g}{2L}\sin\theta \tag{2.39}$$

Similar to the above analysis, note that the horizontal acceleration at detachment is zero, i.e. $\ddot{x} = 0$. So we have, similar to that obtained in Method II,

$$\ddot{x} = -\frac{1}{2}l\sin\theta\dot{\theta}^2 + \frac{1}{2}l\cos\theta\ddot{\theta} = 0 \tag{2.40}$$

giving

$$\ddot{x} = \tan\theta\dot{\theta}^2 \tag{2.41}$$

Using Equations 2.38, 2.39 and 2.41 gives the condition

$$\cos\theta = \frac{2}{3}\cos\theta_0 \tag{2.28}$$

as found in Method I and II.

Method IV- Force Analysis There are other ways to obtain the equation of motion 2.39. One such way is to analyse the forces and torques acting on the system (about the centre of mass[6]).

[6] You can ignore the fact that the system is accelerating in considering torques about the COM - the fictitious force acts on the COM!

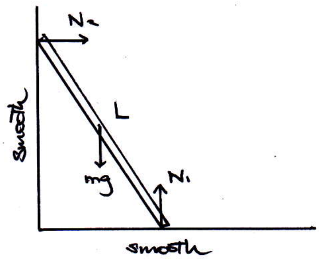


Figure 2.22: Force diagram of ladder for all smooth surfaces.

Consider the force diagram in Figure 2.22. Using Newton's Second Law:

$$\begin{cases} N_1 - mg = m\ddot{y} \\ N_2 = m\ddot{x} \\ \frac{L}{2}N_1 \sin\theta - \frac{L}{2}N_2\cos\theta = I\ddot{\theta} = \frac{1}{12}mL^2\ddot{\theta} \end{cases} \tag{2.42}$$

Eliminating unknowns give

$$g\sin\theta + \ddot{y}\sin\theta - \ddot{x}\cos\theta = \frac{1}{6}L\ddot{\theta} \tag{2.43}$$

Now using the coordinates of the centre of mass $(x, y) = (\frac{L}{2}\sin\theta, \frac{L}{2}\cos\theta)$, we have:

$$\begin{cases} \ddot{x} = \frac{1}{2}L\ddot{\theta}\cos\theta - \frac{1}{2}L\dot{\theta}^2\sin\theta \\ \ddot{y} = -\frac{1}{2}L\ddot{\theta}\sin\theta - \frac{1}{2}L\dot{\theta}^2\cos\theta \end{cases} \tag{2.44}$$

Substituting into Equation 2.43 gives

$$\ddot{\theta} = \frac{3g}{2L}\sin\theta \tag{2.39}$$

as obtained in Method III. Proceed with another steps in Method III to obtain the final answer.

Method V - Lagrangian Mechanics More commonly used in the field is the *Lagrangian Method*. We write the Lagrangian L as:

$$L = T - V \tag{2.45}$$

where T and V are the kinetic and potential energy respectively. For this system, we can write:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mgL\cos\theta \tag{2.46}$$

Using Euler-Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \tag{2.47}$$

we have

$$\ddot{\theta} = \frac{3g}{2L}\sin\theta \tag{2.39}$$

as obtained in Method III.

2.3 Take-home skills

The skills that you should now know is:

1. Conditions for equilibrium
2. Moving the reaction force in stability questions.
3. The importance of counting degrees of freedoms in questions.
4. The energy method

2.4 Response Comments

General Comments

1. Q5: Generally well-answered. The reason why people get this question wrong is typically because of the confusing choice of words I have used.

2. Q6: This is quite hard on itself, but I think I have given enough hints to allow most people to successfully finish the question.

3. Q7: This is generally really hard. I actually did not calculate the positions of the centre-of-mass. but most people who successfully computed the answer used that method. I have a slight worry that this might have been a canonical solution on some webpages (this is a canonical question after all) and people simply copied it down but I think reading these solutions is still helpful for you to learn more on the subject so I am fine with that.

4. Q8: Most people finished (a) and (b) successfully. (b) requires to use of a proof for the angle that the string makes with the horizontal and S4 students are more likely to successfully argue that using circle geometry. YA trigonometric argument is also permissible. (c) is a very hard one and no-one got close to it. This is perhaps something that requires a deep intuition on the subject and requires complete knowledge on "degree of freedom".

5. Q9: This is a complete disaster. Only one person successfully finished the question as he had seen the question in a test in PEP before. S4 students mostly chose Method II which is very difficult when Method I is mathematically straight-forward. Some entirely skipped the question.

6. Overall: The stability condition is better understood by S4 students than S3 students. However, sometimes the physical intuition on that is still lacking and perhaps this is a useful topic to dig into in their own time.

(Updated by Lucas on 7$^{\text{th}}$ August, 2020)

Chapter 3

Non-inertial Frames

In the real world things are not usually in equilibrium. In the classical study of dynamical behaviour of masses, we use the *Newton's Laws of Motion*, which most of you should vividly recall from your elementary mechanics courses. That is concisely stated out here.

Law 3.1. *Newton's Laws of Motion state the following.*

(I) If no net force acts on a body it will move in a straight line at constant speed (or remain at rest if originally at rest).

(II) The net force and acceleration of an object is proportionally related.

$$F_{net} \propto a$$

(III) For every force exerted by an object A on B, there is a force acting on B by A that is equal and opposite.

$$\mathbf{F_{A \to B}} = -\mathbf{F_{B \to A}}$$

A reference frame is a reference system in observing a motion in physics. Note that not all frames of references imply that (I) is true. We therefore define a set of reference frames as *inertial frames* as follows.

Definition 3.1. *There exists a class of preferred reference systems called inertial frames which Newton's First Law holds.*

Frames that do not belong to this set are called *non-inertial frames*. In this exercise, we will try and talk about how to analyse systems in a non-inertial frame. In classical mechanics, we use the *Galilean Transformation* in transforming between inertial frames of reference.

Definition 3.2. *(Galilean Transformation) Suppose we have two inertial frames S and S' in standard configuration, i.e. their origins coincide at $t = t' = 0$ and S' moves with velocity v with respect to S in the x-direction. Then the coordinates of an event P transforms as follows:*

$$\begin{cases} t' = t \\ x' = x - vt \\ y' = y \\ z' = z \end{cases} \tag{3.1}$$

The reverse transform is obtained by substituting v by $-v$. This is known as the ***Galilean Transformation****.*

Transformations in Newtonian mechanics do not preserve acceleration[1] across frames, nor does it regard acceleration as an absolute quantity[2]. So, in transforming from an inertial frame to a non-inertial frame, we require a fictitious force. Here I will teach you how to generate this fictitious force in one of the examples. You are encouraged to then use the concept to answer the remaining problems.

[1] This is to say that acceleration is not invariant in Newtonian Mechanics.

[2] This means that all observers agree upon whether a body is accelerating. This is not true in Special Relativity, where acceleration is absolute across all frames.

3.1 Questions

Question 10: Let us consider a ball hung on the top of the ceiling of a train by an ideal string. The train is accelerating with a constant acceleration a. The angle that the string makes with the vertical is θ (Figure 3.1). We wish to find the relationship between the angle and the acceleration.

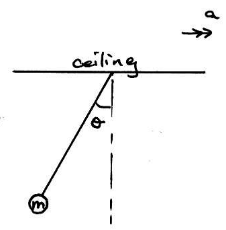


Figure 3.1: A ball hanging on the top of the ceiling of an accelerating train.

(a) Analyse the problem using Newtonian Mechanics in an inertial frame that is simultaneously travelling at the same speed with the train. You should first draw a correct force diagram, then use Newton's Second Law in some directions of your choice.

(b) Consider a general inertial frame S_0 and a non-inertial frame S. Suppose in frame S_0 we have $m\ddot{\mathbf{r}}_{\mathbf{0}} = \mathbf{F}$, where $\mathbf{r}_{\mathbf{0}}$ is the position vector[3] of a particle in S_0. Suppose the position vector $\mathbf{r}$ in frame S is related to that of frame S_0 by

$$\mathbf{r} = \mathbf{r}_{\mathbf{0}} - \mathbf{R}$$

where $\mathbf{R}$ is a general vector. Find the acceleration of the particle in frame S. Hence show that the actual force in the non-inertial frame S includes the actual force $\mathbf{F}$ and a fictitious force. Explicitly show the general form of a fictitious for a non-inertial frame.

(c) Hence for the special case $\ddot{\mathbf{R}} = 0$, find the apparent force in S. What does this case represent?

(d) Solve the problem in the first part by analysing inside the accelerating frame of the train.

Question 11: Two blocks are stacked on top of each other on a frictionless plane. The upper block has mass m and is lying at the edge of the lower block of mass M. The static and kinetic coefficients of friction between the masses are μ_s and μ_k respectively. A horizontal force F is exerted on the upper block (Figure 3.2).

[3] A position vector in frame S is a vector that denotes the position of the particle in a reference frame. The vector is defined to be drawn from the origin of S to the position P of the particle in frame S.

(a) Draw the force diagrams of the blocks.

(b) The horizontal force is increased slowly from 0. Find the maximum force F_{lim} such that there is no relative motion between the blocks. Find the acceleration fo the blocks.

(c) When the force exceeds the limiting value F_{lim} in the previous part, describe what occurs. Suppose the upper block is acted by a horizontal force $F > F_{lim}$. By analysing in the lower block's frame or otherwise, calculate the relative acceleration of the upper block compared to the lower block. If the upper block has a negligible dimension, find the time required for the block to travel across the lower block of length L.

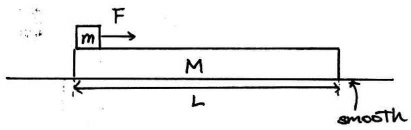


Figure 3.2: Two blocks stacked on top of each other.

Question 12: A block of mass m is lying on a wedge of mass M with an angle θ. All surfaces are smooth and the wedge is not fixed to the ground (Figure 3.3). Analyse the problem in wedge's frame of reference.

(a) Draw the force diagram of the block in the frame of the wedge.

(b) Compute the acceleration of the block in this frame. If the block is initially of height H above the table, find the time required for the block (of negligible dimension) to reach the bottom of the wedge.

(c) Repeat your analysis in the inertial frame of the lab. Again find the relative acceleration between the two objects (You should find out how much easier the analysis is in the non-inertial frame!).

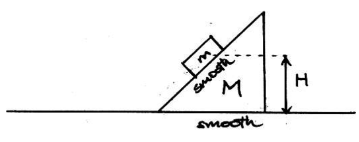


Figure 3.3: The block and the movable wedge.

Question 13: A pendulum with a bob of mass m, an ideal string of length l is attached to a cart of mass M free to travel in the upper smooth plane (Figure 3.4). Find the small angle oscillation frequency of the system. You may assume that $\dot{\theta}^2 \ll \frac{(M+m)g}{ml}$.[*Hint: Consider the problem in the frame of the cart.*]

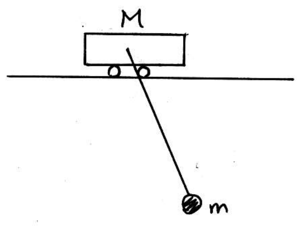


Figure 3.4: A pendulum with movable base.

Question 14: This question is about pulleys (exciting!).

(a) Two masses m_1 and m_2 are connected by an ideal string to a smooth ideal pulley (Figure 3.5). Find the tension in the string and the acceleration of the two masses respectively.

(b) Now consider three masses m_1, m_2 and m_3 connected to two pulleys by ideal strings in the configuration shown in Figure 3.6.

(i) First analyse the system in the inertial frame of reference of a person setting up the system (Lab frame). Find the accelerations of the three masses.

(ii) Now treat the the lower pulley system as a box with effective mass M. Using the results in (a), find the acceleration a of the lower pulley system in terms of M, m_1 and g.

(iii) Analysing in the non-inertial frame of reference of the pulley system, obtain an expression for the acceleration a' of the two lower masses m_2 and m_3 in terms of a, g, m_2 and m_3.

(iv) Find the tension of the string connecting m_1 to the lower pulley system in terms of the tension in the string **in** the lower pulley system.

(v) Hence, obtain the effective mass of the lower pulley system. Check your answer using the result in (a) and (b)(i).

(c) Consider now the infinite pulley system shown in Figure 3.7, where all masses have the same mass m ($m_i = m$, $i \in \mathbb{N}$).

(i) Obtain a recursion relation for the effective mass at stage n (n^{th} pulley up the lowest one), where n is a positive integer.

(ii) What is the limit when $n \to \infty$?

(iii) Hence, find the effective mass of the infinite pulley system for the highest mass. Calculate the acceleration of it.

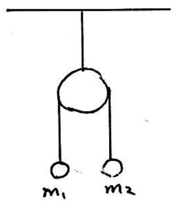


Figure 3.5: Two masses m_1 and m_2 are connected by an ideal string to a smooth ideal pulley.

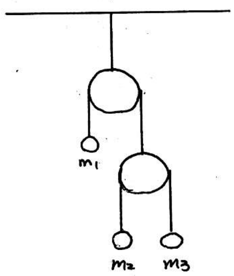


Figure 3.6: The 2-step pulley system.

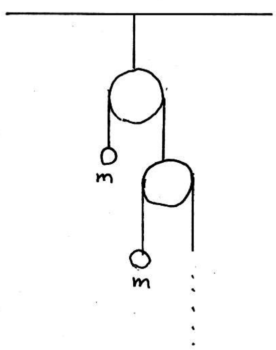


Figure 3.7: The Infinite pulley system.

3.2 Solutions and Commentary

Question 10: Consider a ball hanged on the top of the ceiling of a uniformly accelerating train by an ideal string. Find the relationship between the angle that the string makes with the vertical θ and the acceleration a of the train.
Solution: We analyse the problem with two different methods.

Inertial Frame First we analyse the problem in the inertial frame that has a velocity instantaneously equal to the velocity of the train. We call this the instantaneous rest frame (IRF). Drawing a force diagram of the system:

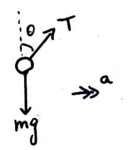


Figure 3.8: Force diagram of the ball in the IRF.

and using Newton's Second Law:

$$\begin{cases} T\cos\theta = mg \\ T\sin\theta = ma \end{cases} \tag{3.2}$$

We hence have

$$a = g\tan\theta \tag{3.3}$$

which gives the relationship we wish to find.

Deriving the fictitious force Before we analyse the problem in the non-inertial frame of the train, we must first understand what we have to change in observing the problem in a general linearly accelerating non-inertial frame. Consider a general inertial frame S_0 and a non-inertial frame S. Suppose in frame S_0 we have $m\ddot{\mathbf{r}}_\mathbf{0} = \mathbf{F_0}$, where $\mathbf{r_0}$ is the position vector of a particle in S_0. Let us suppose that the position vector $\mathbf{r}$ in frame S is related to that of frame S_0 by

$$\mathbf{r} = \mathbf{r_0} - \mathbf{R} \tag{3.4}$$

where $\mathbf{R}$ is a general vector. Then differentiating Equation 3.4 twice gives:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_\mathbf{0} - \ddot{\mathbf{R}} \tag{3.5}$$

so multiplying both sides by m gives:

$$m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}_\mathbf{0} - m\ddot{\mathbf{R}} \tag{3.6}$$

$$\mathbf{F} = \mathbf{F_0} - m\ddot{\mathbf{R}} \tag{3.7}$$

We have found a relationship that links the forces in the two frames: The force experienced by particle in the non-inertial frame S (LHS of Equation 3.7) is equal to the force experienced in the inertial frame S_0 plus an additional term $-m\ddot{\mathbf{R}}$. This term is known as the fictitious force (a.k.a. pseudo-force) in the non inertial frame.

Non-inertial Frame With this in mind we can draw a force diagram again, this time in the accelerating frame of the train. Noting that the fictitious force points to the opposite direction of the acceleration in the inertial frame, we have:

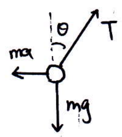


Figure 3.9: Force diagram of the ball in the accelerating frame of the train.

The ball is in equilibrium in this frame so we have

$$\begin{cases} T\cos\theta = mg \\ T\sin\theta = ma \end{cases} \tag{3.8}$$

same as before. We hence have

$$a = g\tan\theta \tag{3.3}$$

same as before.

Question 11: Two blocks are stacked on top of each other on a frictionless plane. The upper block has mass m and is lying at the edge of the lower block of mass M. The static and kinematic coefficients of friction between the masses are μ_s and μ_k respectively. A horizontal force F is exerted on the upper block (Figure 3.2).

(a) The horizontal force is increased slowly from 0. Find the maximum force F_{lim} such that there is no relative motion between the blocks. Find the acceleration of the blocks.

(b) Suppose the upper block is acted by a horizontal force $F > F_{lim}$. By analysing in the lower block's frame or otherwise, calculate the relative acceleration of the upper block compared to the lower block. If the upper block has a negligible dimension, find the time required for the block to travel across the lower block of length L.

Solution:

No Relative Motion Suppose first that there is no relative motion between the blocks. The force diagrams of the two blocks are:

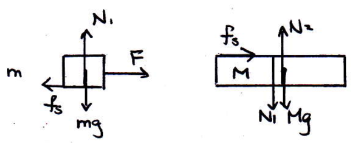


Figure 3.10: Force diagrams of the two blocks when there is no relative motion between the blocks.

Considering the two blocks together as the system and using Newton's Second Law:

$$F = (m + M)a \tag{3.9}$$

This gives

$$a = \frac{F}{m + M} \tag{3.10}$$

Now consider the lower block. This gives,

$$fs = Ma = \frac{FM}{m + M} \leq \mu_s N \tag{3.11}$$

Rearranging hence gives

$$F \leq \frac{\mu_s m(m + M)g}{M} \tag{3.12}$$

Relative Motion Now suppose there is relative motion between the blocks. The force diagrams for the two blocks are:

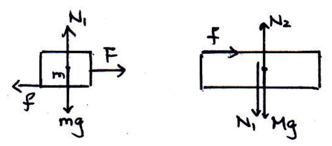


Figure 3.11: Force diagrams of the two blocks.

The standard method is to analyse the forces of the two blocks. The kinetic friction is given by

$$f = Ma_m = \mu_k mg \tag{3.13}$$

Rearranging gives the acceleration of the lower block a_M:

$$a_M = \frac{\mu_k mg}{M} \tag{3.14}$$

Newton's Second Law on the horizontal forces of the upper block gives:

$$ma_m = F - f \tag{3.15}$$

$$a_m = \frac{F - \mu_k mg}{m} \tag{3.16}$$

The relative acceleration is hence,

$$\Delta a = a_m - a_M = \frac{F}{m} - \mu_k mg(\frac{1}{M} + \frac{1}{m}) \tag{3.17}$$

Alternatively one can use the non-inertial frame of the lower block. The force diagram of the upper block in this frame gives:

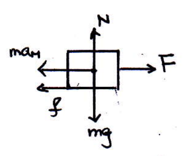


Figure 3.12: Force diagrams of the upper block in the lower block's frame.

Using Newton's Second Law gives:

$$F - f - ma_M = ma' \tag{3.18}$$

where a' is the relative acceleration between the two. Using Equations 3.13 and 3.14 gives the same answer as before:

$$a' = \frac{F}{m} - \mu_k mg(\frac{1}{M} + \frac{1}{m}) \tag{3.19}$$

where a' and Δa means exactly the same thing. Using simple kinematics, we have that

$$L = \frac{1}{2}\Delta a t^2 \tag{3.20}$$

Thus

$$t = \sqrt{2L}(\frac{F}{m} - \mu_k mg(\frac{1}{M} + \frac{1}{m}))^{-\frac{1}{2}} \tag{3.21}$$

Question 12: A block of mass m is lying on a wedge of mass M with an angle θ. All surfaces are smooth and the wedge is not fixed to the ground (Figure 3.3). Find the relative acceleration between the two objects. If the block is initially of height H above the table, find the time required for the block (of negligible dimension) to reach the bottom of the wedge.

Solution: This problem is most easily solved in the frame of the wedge. The force diagrams of the two objects are:

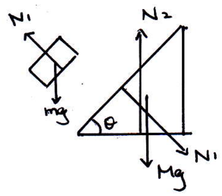


Figure 3.13: Force diagrams of the block and the wedge.

Applying Newton's Second Law to the wedge gives

$$\begin{cases} N_1 \sin\theta = Ma \\ N_2 = N_1 \cos\theta + Mg \end{cases} \tag{3.22}$$

Consider the force diagram of the block in the accelerating frame of the wedge[4]:

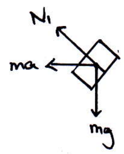


Figure 3.14: Force diagrams of the block in the non-inertial frame of the wedge.

Newton's Second Law gives:

$$\begin{cases} N_1 + \frac{m}{M} N_1 \sin^2\theta = mg\cos\theta \\ ma' = \frac{m}{M} N_1 \sin\theta\cos\theta + mg\sin\theta \end{cases} \tag{3.23}$$

[4]Note here that the fictitious force acts on the centre of mass of the object.

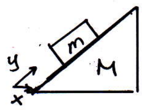


Figure 3.15: Defining the coordinates for Lagrangian analysis.

The first equation gives:

$$N_1 = \frac{mg\cos\theta}{1+\frac{m}{M}\sin^2\theta} \tag{3.24}$$

Substituting this into the second equation in Equation 3.23 gives:

$$ma = \frac{m}{M}\left(\frac{mg\cos\theta}{1+\frac{m}{M}\sin^2\theta}\right)\sin\theta\cos\theta + mg\sin\theta \tag{3.25}$$

Simplifying gives

$$a' = g\sin\theta\left(1+\frac{\cos^2\theta}{\frac{M}{m}+\sin^2\theta}\right) \tag{3.26}$$

Lagrangian Method Defining the coordinates as in the following diagram. The kinetic energy and potential energy is:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\dot{y}\cos\theta)^2 + \frac{1}{2}m\dot{y}^2\sin^2\theta \tag{3.27}$$

$$V = -mgy\sin\theta \tag{3.28}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y}\cos\theta + \frac{1}{2}m\dot{y}^2 \tag{3.29}$$

The Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{3.30}$$

where q_i are the general coordinates of the system. Then since $\frac{\partial L}{\partial x}$ is zero, the conjugate momentum of the x coordinate (momentum in x-direction) is conserved. We can write:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + m\ddot{y}\cos\theta = 0 \tag{3.31}$$

giving

$$\ddot{x} = -\frac{m\cos\theta}{M+m}\ddot{y} \tag{3.32}$$

For the y-coordinate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{x}\cos\theta + m\dot{y}) = m\ddot{x}\cos\theta + m\ddot{y} \tag{3.33}$$

$$\frac{\partial L}{\partial y} = mg\sin\theta \tag{3.34}$$

This gives

$$\ddot{x}\cos\theta + \ddot{y} = g\sin\theta \tag{3.35}$$

So using Equation 3.32 gives

$$(1 - \frac{m\cos^2\theta}{m+M})\ddot{y} = g\sin\theta \tag{3.36}$$

which evaluates to

$$\ddot{y} = g\sin\theta(1 + \frac{\cos^2\theta}{\frac{M}{m} + \sin^2\theta}) \tag{3.37}$$

same as the above method.

Time Required The time required for the block to travel a distance $\frac{H}{\sin\theta}$ is

$$t = \sqrt{\frac{2H}{a'\sin\theta}} \tag{3.38}$$

with a' being the relative acceleration found in Equation 3.26.

Question 13: A pendulum with a bob of mass m, an ideal string of length l is attached to a cart of mass M free to travel in the upper smooth plane (Figure 3.4). Find the small angle oscillation frequency of the system. You may assume that $\dot{\theta}^2 \ll \frac{(M+m)g}{ml}$.
Solution: Draw a force diagram of the pendulum in the frame of the cart.

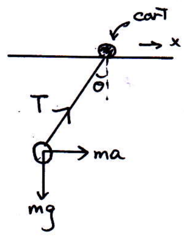


Figure 3.16: Force diagram of the bob in the non-inertial frame of the cart.

The acceleration of the cart is found from analysing Newton's Second Law on the cart:

$$T \sin\theta = Ma \tag{3.39}$$

$$a = \frac{T\sin\theta}{M} \tag{3.40}$$

So the fictitious force has a magnitude:

$$F_f = ma = \frac{m}{M} T \sin\theta \tag{3.41}$$

Now the restoring force is

$$m\ddot{x} = -T\sin\theta(1 + \frac{m}{M}) \tag{3.42}$$

Using small angle approximation $\theta \approx \sin\theta \approx \tan\theta$, and the fact that $T\cos\theta \approx mg$, we have

$$m\ddot{x} = -\frac{x}{l} mg \sin\theta(1 + \frac{m}{M}) \tag{3.43}$$

So the frequency of oscillation can be obtained by comparing with the simple spring-mass system $m\ddot{x} = -kx$:

$$\omega_0^2 = \frac{mg}{l}(\frac{1}{m} + \frac{1}{M}) \tag{3.44}$$

Lagrangian Analysis Define the x-coordinate of the cart as the coordinate of the system (see Figure 3.16 for the definition for the coordinates). The kinetic and potential energies are:

$$T = \frac{1}{2}m(\dot{x} + l\dot{\theta})^2 + \frac{1}{2}ml^2\dot{\theta}^2\sin^2\theta + \frac{1}{2}M\dot{x}^2 \tag{3.45}$$

$$V = -mgl\cos\theta \tag{3.46}$$

so the Lagrangian is:

$$L = \frac{1}{2}(m_M)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{3.47}$$

Again using Euler-Lagrange Equation (Equation 3.30), for the x-coordinate:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0 \tag{3.48}$$

Rearranging gives

$$\ddot{x} = ml\frac{\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta}{m+M} \tag{3.49}$$

This shows that momentum is conserved in this direction. Now consider θ:

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta} \tag{3.50}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta \tag{3.51}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta} \tag{3.52}$$

This gives:

$$\ddot{x}\cos\theta + l\ddot{\theta} = -g\sin\theta \tag{3.53}$$

Now using Equation 3.49 and 3.53, and expanding the expression to first order in θ, we have:

$$ml(\frac{\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta}{m+M})\cos\theta + l\ddot{\theta} = -g\sin\theta \tag{3.54}$$

$$\ddot{\theta} = -\frac{g}{l}\theta - \frac{m}{M+m}(\dot{\theta}^2\theta - \ddot{\theta}) \tag{3.55}$$

This can be rewritten into

$$\ddot{\theta} = -(\frac{m}{M}+1)(\frac{g}{l} + \frac{m}{m+M}\dot{\theta}^2)\theta \tag{3.56}$$

This is not an SHM. However, when the angular velocity is sufficiently small, i.e. $\dot{\theta}^2 \ll \frac{(M+m)g}{ml}$, Equation 3.56 to

$$\ddot{\theta} = -(\frac{m}{M}+1)\frac{g}{l}\theta \tag{3.57}$$

so the frequency of oscillation is

$$\omega_0^2 = \frac{mg}{l}(\frac{1}{m} + \frac{1}{M}) \tag{3.44}$$

as before.

Question 14: A question on Atwood Machines.

(a) Two masses m_1 and m_2 are connected by an ideal string to a smooth ideal pulley (Figure 3.5). Find the tension in the string and the acceleration of the two masses respectively.

(b) Now consider three masses m_1, m_2 and m_3 connected to two pulleys by ideal strings in the configuration shown in Figure 3.6. Find the accelerations of the three masses. Find the effective mass of the lower pulley system that induces such acceleration.

(c) Consider now the infinite pulley system shown in Figure 3.7, where all masses have the same mass m ($m_i = m$, $i \in \mathbb{N}$). Calculate the acceleration of the uppermost ball due to the entire pulley system.

Solution:

Simple Atwood Machine The force diagrams of the masses are:

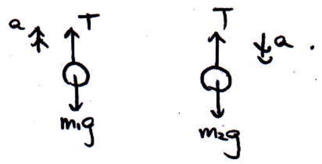


Figure 3.17: The force diagrams of a Simple Atwood Machine.

Without loss of generality assume that $m_1 < m_2$. Let the acceleration of the masses be a. Then Newton's Second Law gives:

$$\begin{cases} m_2 g - T = m_2 a \\ T - m_1 g = m_1 a \end{cases} \tag{3.58}$$

Solving for T and a gives

$$a = \frac{m_2 - m_1}{m_1 + m_2} g \tag{3.59}$$

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g \tag{3.60}$$

Compound Atwood Machine Now consider two Atwood Machines linked together as illustrated in Figure 3.6. There are two methods to solve the system. The most straightforward one is the write down all the equations. Let us define a_i be the acceleration of the i^{th} mass and take upwards to be the positive direction. Since the pulleys are massless, there must be no net force on it, otherwise there will be infinite acceleration resulted on the object. The tension in the upper pulley system is therefore twice that in the lower pulley system. Newton's Second Law gives:

$$\begin{cases} 2T - m_1 g = m_1 a_1 \\ T - m_2 g = m_2 a_2 \\ T - m_3 g = m_3 a_3 \end{cases} \tag{3.61}$$

The total length of the string is conserved. The accelerations of the three masses are related by[5]

$$a_1 = -\frac{a_2 + a_3}{2} \tag{3.62}$$

This evaluates to give

$$a_1 = g\frac{4m_2m_3 - m_1(m_2 + m_3)}{4m_2m_3 + m_1(m_2 + m_3)} \tag{3.63}$$

$$a_2 = -g\frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2 + m_3)} \tag{3.64}$$

$$a_3 = -g\frac{4m_2m_3 + m_1(m_3 - 3m_2)}{4m_2m_3 + m_1(m_2 + m_3)} \tag{3.65}$$

which a_1 can be written as

$$a_1 = g\frac{\frac{4m_2m_3}{m_2+m_3} - m_1}{\frac{4m_2m_3}{m_2+m_3} + m_1} \tag{3.66}$$

nicely put in the form of Equation 3.59. So the effective mass is

$$m_{\text{eff}} = \frac{4m_2m_3}{m_2 + m_3} \tag{3.67}$$

An alternative method is to consider the two masses in the lower pulley system. The force diagrams for two balls in this frame are:

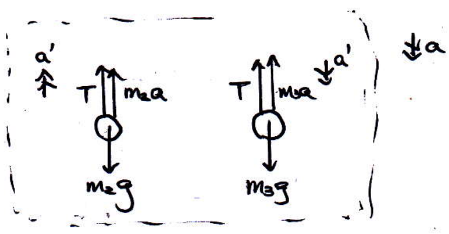


Figure 3.18: The force diagrams of the lower masses in the accelerating frame of the lower pulley system.

[5] a_2 and a_3 are typically pointing in different directions. Their sum gets rid of the "relative acceleration" with respect to the lower pulley, leaving the term that gives the acceleration of the entire pulley system. Since both fo the masses are accelerating at this acceleration, this number is doubled (Hence the $\frac{1}{2}$ factor). The final negative sign comes from the opposite direction of the acceleration of m_1 and the lower pulley system.

Again assume without loss of generality assume that $m_2 < m_3$. Then

$$\begin{cases} m_3 g - T - m_3 a = m_3 a' \\ m_2 g - T - m_2 a = m_2 a' \end{cases} \tag{3.68}$$

where a is the acceleration of the lower pulley system (downwards) and a' is the acceleration of the masses in this frame. Solving for T gives

$$T = \frac{2m_2 m_3}{m_2 + m_3}(g - a) \tag{3.69}$$

The tension in the upper string (by the argument in Method I) is $2T$:

$$2T = \frac{4m_2 m_3}{m_2 + m_3}(g - a) \tag{3.70}$$

Let us write

$$m_{\text{eff}} = \frac{4m_2 m_3}{m_2 + m_3} \tag{3.67}$$

Considering Newton's Second Law on the upper mass gives:

$$2T - m_1 g = m_1 a \tag{3.71}$$

Using Equations 3.67 and 3.71 give

$$m_1 g + m_1 a = m_{\text{eff}}(g - a) \tag{3.72}$$

Hence giving

$$a = \frac{m_{\text{eff}} - m_1}{m_1 + m_{\text{eff}}} \tag{3.73}$$

as obtained before. You can obtain the other accelerations by converting them back to lab frame quantities.

Infinite Atwood Machine This seems like a really difficult question but the answer is extremely easy. Note that the system is an extension to the Compound Atwood Machine when $m_1 = m_2 = m_3$. From previous results we know that we can treat the lower as having an effective mass of

$$m_{\text{eff}} = \frac{4m_2 m_3}{m_2 + m_3} \tag{3.67}$$

Let us consider some system in this Atwood Machine. The effective mass must be

$$M_{\text{eff}} = \frac{4mx}{m + x} \tag{3.74}$$

where x is the effective mass of the system just lower than the one in consideration. This allows us to set up a recurrence relation:

$$a_{n+1} = \frac{4ma_n}{m + a_n} \tag{3.75}$$

What happens when the number of system goes to infinity? You can check that the recurrence relation implies a convergent series, but let's suppose we assume that the result is convergent. Then when infinity is reached we must have

$$a_{n+1} = a_n = \frac{4ma_n}{m + a_n} \tag{3.76}$$

This allows us to solve a quadratic which gives

$$a_n = 3m \tag{3.77}$$

Substituting into Equation 3.73 gives

$$a = \frac{g}{2} \tag{3.78}$$

An alternative (cheaty) method is to note that the tension in the string is proportional to the acceleration of the systems[6]. The topmost system lives in a world of acceleration g, the second one $g - a$. We know that the tension in the string of the upper system is twice that of the one that is just under it. So

$$\frac{2T}{T} = \frac{g}{g - a} \tag{3.79}$$

This solves to give

$$a = \frac{g}{2} \tag{3.78}$$

as obtained above.

3.3 Take-home skills

The skills that you should now know are:

1. How to use of non-inertial frames in problems.
2. How to deal with recurrence relations.

[6]You can think about this - if you live in a world which the acceleration is multiplied by a factor of α, the tension in the string, by Equation 3.60, must also be multiplied by a factor of α

3.4 Response Comments

General Comments

1. This is in fact one of the worst performed exercises. This is mostly attributed to the fact that the last question on Atwood machines is poorly performed.

2. Q10: I have initially asked people to finish this exercise before the lecture on reference frames. Only 2 people handed in their responses, and 1 of them misunderstood the question owing to the choice of wording in the question. It is originally intended that this would allow students to think about how to set up a problem in a linearly accelerating frame but few are able to do it. It is therefore suggested to just put the question in the introduction of the chapter.

3. Q11: Most people are able to do the question. Here analysing in the inertial frame might be easier and some students got away with using that.

4. Q12: Some entirely missed the fact that the wedge is not fixed. Others drawn the fictitious forces incorrectly. Only one student provided a correct response.

5. Q13: This is one fo the most straightforward questions. It was intended that the oscillation amplitude be small and students would overlook the fact that there are several approximations taken in. This is explained in the example class.

6. Q14: Truly horrifying. Most people who got this far successfully completed (a) which is very straightforward. Almost every one either treated the second pulley system in (b) as having an effective mass of $m_2 + m_3$ or assuming that the acceleration of the second mass and the third mass in the second pulley system in the inertial frame is equal in magnitude. This resulted in everyone almost losing ~ 25 marks there.

7. Summary: This is one of the most important skills in force analysis and students should definitely revisit these topics if they can in the future given the poor performance of the exercise.

(Updated by Lucas on 9^{th} August, 2020)

Chapter 4

Projectile Motion

One important topic in HKPhO that we looked over in the lectures is the analysis of projectile motion of particles. This is essentially a kinematic question in two-dimensions. The only thing to remember is that there is no acceleration in the direction orthogonal to the local gravitational field **g**. Here are just some questions you can practice on.

4.1 Questions

Question 15: A particle is ejected with an initial speed v_0 with an angle θ to the horizontal.

Figure 4.1: Sketch of some quantities the motion of the particle.

(a) Derive the equation of trajectory:

$$y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta} \tag{4.1}$$

(b) Find the maximum height the particle can reach compared to the point where it is ejected (see Figure 4.1). Calculate the time for the particle to reach this position.

(c) Now suppose the same particle is put on a smooth incline of angle ϕ. The particle is ejected with the same initial velocity v_0 and θ to the horizontal. Find the equation of trajectory in this case.

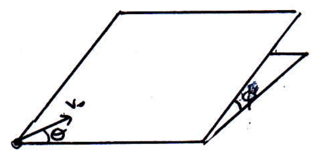


Figure 4.2: Sketch of the particle on the smooth incline of angle ϕ.

Question 16: Consider a ball being shot down a cliff of incline angle ϕ. The initial velocity for the ball is u and the angle from the horizontal is θ.

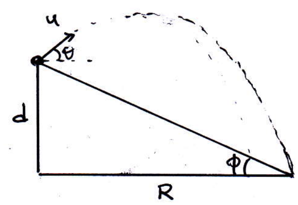


Figure 4.3: A ball being shot down a cliff of angle ϕ.

There are two approaches to the problem you may choose.

(I) HKPhO - Trigonometry

(a) The equation of trajectory is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \tag{4.2}$$

Write down the relationship between y and x if the ball is to land on the cliff. Hence express R in terms of u, g, θ and ϕ.

(b) Using the compound angle formulae, find an alternative expression for R. Show that R is maximum when $2\theta+\phi = \frac{\pi}{2}$. Hence show the following:

(i) The maximum range is given by

$$R_0^2 + 2dR_0 = R_m^2 \tag{4.3}$$

where $R_0 = \frac{u^2}{g}$, R_m is the maximum horizontal range R and $d = R_m\tan\phi$.

(ii) The quantities are related by:

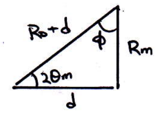


Figure 4.4: A relation between quantities.

(iii) The condition for maximum range R_m is given by

$$d \tan 2\theta_m = R_m \tag{4.4}$$

(II) Calculus

(a) The equation of trajectory is

$$y = x \tan\theta - \frac{gx^2}{2u^2 \cos^2\theta} \tag{4.5}$$

Write down the relationship between y and x if the ball is to land on the cliff. Hence express R in terms of u, g, θ and ϕ.

(b) By differentiating the expression obtained in (a), find an alternative expression for R. Hence show the following:

(i) The condition for maximum range R_m is given by

$$d \tan 2\theta_m = R_m \tag{4.6}$$

(ii) The maximum range is given by

$$R_0^2 + 2dR_0 = R_m^2 \tag{4.7}$$

where $R_0 = \frac{u^2}{g}$, R_m is the maximum horizontal range R and $d = R_m \tan\phi$.

(iii) The quantities are related by Figure 4.4.

Question 17: What is the maximum angle to the horizontal at which a stone can be thrown and be always moving away from the observer?

Question 18: An explosion in Beirut occurred on 5 August, 2020. The cause for the explosion is initially attributed to the ignition of a large amount of ammonium nitrate. At least 100 people were killed. Here we objectively analyse the explosion and propose some possible safety measures.

(a) There are typically 3 main reasons why people are injured from explosions.

   a) Mechanical injury from the initial shockwave.
   b) Injuries from flying debris/ objects caused by the explosion.
   c) Physically colliding with objects owing to explosion shockwave.

   Let us suppose a normal sized-human can sustain up to E_0 units of energy. The energy released from the explosion is E_e. The person is initially standing at a distance R from the centre of the explosion (Figure 4.5).

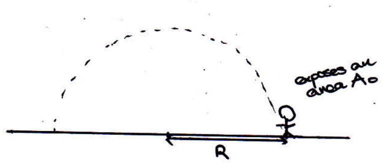


Figure 4.5: Configuration at the moment of explosion. The man is standing at a distance R from the centre of the explosion.

   Suppose all the energy from the explosion is uniformly released as a shock wave in the upper hemisphere from the centre. If the person facing the centre exposes an area of A_0 of his body, find the energy absorbed by the human. You may assume that locally the energy from the shockwave will be entirely absorbed by the object the wave comes into contact with. Hence find the condition for which he survives the blast.

(b) Now account for the flying debris. You can suppose that the debris is uniformly released from the centre of the explosion, each with a mass of dm and a velocity of u.

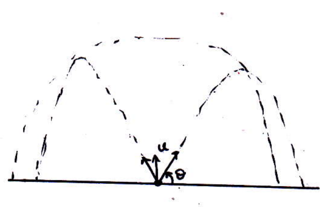


Figure 4.6: Flying debris envelope.

(i) Find the a relation relating the coordinates of the particle (x, y) and the initial release angle θ.

(ii) Obtain a quadratic equation in $\tan\theta$. For $\tan\theta$ to exist, find the minimum value of u in terms of g and (x, y).

(iii) Find the corresponding value for $\tan\theta$ for the minimum u in (ii).

(iv) Hence obtain an equation for the envelope of the explosion. This gives an "explosion" boundary which contains all the possible trajectories for fixed u (see Figure 4.6).

(c) Suppose this person is outside the envelope described in (b) and Figure 4.6. Model a person as a cube of side L. Suppose that the person takes up all the kinetic energy from the shockwave which the body has contact with (as in (a)). He flies backwards and hits a car with his back.

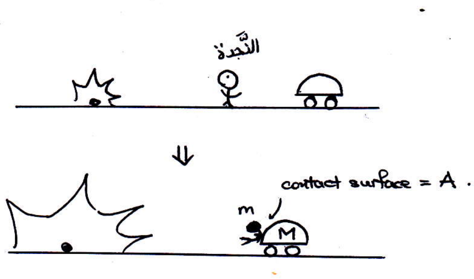


Figure 4.7: Schematic for flying man.

Suppose $m < M$. If they are free to move only in a straight line, find the change in momentum of the man in the collision if the collision is elastic. Given that the contact surface area is A and the time of collision is Δt, find the average pressure on the human. Describe his possible injuries.

(d) Suggest some ways of minimising injuries from the explosion. Also suggest something that will improve our modelling the system.

Question 19: *(Extremely Difficult, need Calculus)* A particle is thrown with initial velocity u from the ground. The angle to horizontal at launch is θ. The particle path ends when it first hits the ground again. Find the angle of launch which

(a) Maximises the area covered by the particle.

(b) Maximises the path length travelled by the particle.

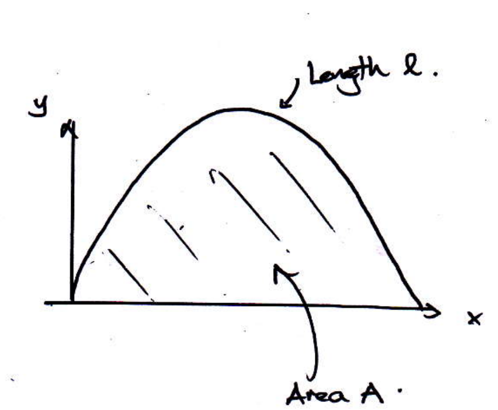


Figure 4.8: Sketch of the particle path and related quantities.

[*Hints:*

1. *This integral is quite useful:*

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log |\sec x + \tan x| \tag{4.8}$$

2. *Plot of graph*

$$x = 2 - \sin\theta \log |\frac{1 + \sin\theta}{1 - \sin\theta}| \tag{4.9}$$

is shown below:

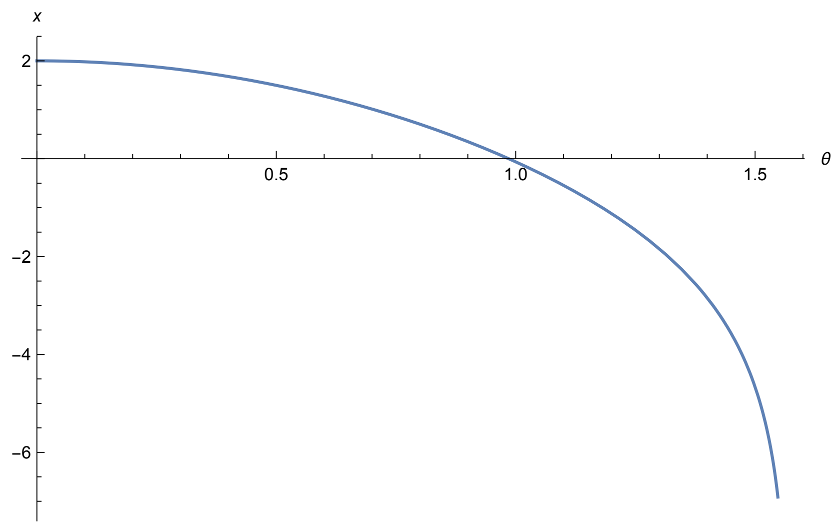


Figure 4.9: Graph of Equation 4.9. Note that θ is in radians.

]

4.2 Solutions and Commentary

Question 15: A particle is ejected with an initial speed v_0 with an angle θ to the horizontal (Figure 4.1). Derive the equation of trajectory and find the maximum height that the particle can reach and the time to reach that point. Find the equation of trajectory if the same particle is put on a smooth incline of angle ϕ. The particle is ejected with the same initial velocity v_0 and θ to the horizontal (Figure 4.2).
Solution: First we drive the equation of trajectory. This is bookwork:

$$\begin{cases} x = v_0 t \cos\theta \\ y = v_0 t \sin\theta - \frac{1}{2} g t^2 \end{cases} \tag{4.10}$$

Eliminating t gives the equation of trajectory.

$$y = x \tan\theta - \frac{g}{2v_0^2 \cos^2\theta} x^2 \tag{4.11}$$

The second part of the question is also bookwork. The maximum height can be found from energy conservation (or a kinematic equation)

$$v_y^2 = 2gH \tag{4.12}$$

where H is the maximum vertical displacement reached by the particle. Rearranging gives:

$$H = \frac{v_0^2 \sin^2\theta}{2g} \tag{4.13}$$

The time required is

$$H = \frac{1}{2} v_0 t \sin\theta \tag{4.14}$$

$$t = \frac{v_0 \sin\theta}{g} \tag{4.15}$$

If the motion occurs on a frictionless plane, replace the quantities $g \to g \sin\phi$. The y' coordinate along the incline is now $y' = \frac{y}{\sin\phi}$, where y is the vertical displacement of the ball.

Question 16: Consider a ball being shot down a cliff of incline angle ϕ. The initial velocity for the ball is u and the angle from the horizontal is θ (Figure 4.3). Denote R as the horizontal displacement for the ball as it first hits the cliff and d be the (downward positive) vertical displacement. Show the following:

1. R is maximum when $2\theta + \phi = \frac{\pi}{2}$.
2. The maximum range is given by
$$R_0^2 + 2dR_0 = R_m^2 \tag{4.16}$$
where $R_0 = \frac{u^2}{g}$, R_m is the maximum horizontal range R and $d = R_m \tan\phi$.
3. The quantities are related as illustrated in Figure 4.4.
4. The condition for maximum range R_m is given by
$$d \tan 2\theta_m = R_m \tag{4.17}$$

Solution: There are two approaches to the problem. I will begin by using the subsidiary angle method.

Method I - Subsidiary Angle We note that elementary trigonometry gives
$$d = R\tan\phi \tag{4.18}$$

Using the equation of trajectory ($u \to v_0$)
$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \tag{4.11}$$

This gives
$$-R\tan\phi = R\tan\theta - \frac{gR^2}{2u^2\cos^2\theta} \tag{4.19}$$
Solving for R and using the double angle formulae gives:
$$R = \frac{u^2}{g}[\sin 2\theta + \tan\phi(\cos 2\theta + 1)] \tag{4.20}$$

Now using the compound angle formula for sine,
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \tag{4.21}$$

Equation 4.20 gives
$$R = \frac{u^2}{g\cos\phi}[\sin(2\theta + \phi) + \sin\phi] \tag{4.22}$$
To maximise R we simply need
$$2\theta + \phi = \frac{\pi}{2} \tag{4.23}$$

Therefore the condition for maximum range, using Equations 4.22 and 4.18 is

$$d \tan 2\theta_m = R \tag{4.24}$$

Using Equation 4.22 we have:

$$R_m = R_0[\cos\phi + \tan\phi(1 + \sin\phi)] \tag{4.25}$$

so proving the result 4.16 is merely just an algebraic matter.

$$R_0^2 + 2dR_0 = R_m^2 \tag{4.16}$$

From this you can show that

$$(R_0 + d)^2 = R_m^2 + d^2 \tag{4.26}$$

so by converse of Pythagoras' Theorem the quantities are related by a right-angled triangle.

Method II - Calculus Most of the algebra in this method is the same as that in Method I. The only difference is, upon reaching Equation 4.20, differentiate the equation to obtain the extrema of R.

$$\frac{\partial R}{\partial \theta} = \frac{2u^2}{g}(\cos 2\theta - \tan\phi \sin 2\theta) = 0 \tag{4.27}$$

So this gives:

$$\tan\phi = \cot 2\theta \tag{4.28}$$

From which (since the angles are physically constrained between 0 and $\frac{\pi}{2}$) it can be concluded that

$$2\theta + \phi = \frac{\pi}{2} \tag{4.23}$$

as obtained before.

Question 17: What is the maximum angle to the horizontal at which a stone can be thrown and be always moving away from the observer?
Solution: The difficult thing here is to figure out the condition required in the question. We want the magnitude to be non-decreasing so

$$\frac{d(r^2)}{dt} \geq 0 \tag{4.29}$$

Equivalently in vectorial notation this is:

$$\mathbf{v} \cdot \mathbf{r} \geq 0 \tag{4.30}$$

So we can write, at the critical angle:

$$(u\cos\theta)(ut\cos\theta) + (u\sin\theta - gt)(ut\sin\theta - \frac{1}{2}gt^2) = 0 \tag{4.31}$$

$$u^2 - \frac{3}{2}gu\sin\theta t + \frac{1}{2}gt^2 = 0 \tag{4.32}$$

This is a quadratic solution in t, which requires real solutions. The determinant hence gives:

$$\frac{9}{4}u^2\sin^2\theta - 4(\frac{1}{2}g^2)u^2 \geq 0 \tag{4.33}$$

From which we obtain

$$\sin^2\theta \geq \frac{8}{9} \tag{4.34}$$

The sine function is monotonically increasing within the physically acceptable range of θ. Therefore the maximum angle is given by:

$$\sin\theta = \frac{2\sqrt{2}}{3} \tag{4.35}$$

Question 18:

(a) There are typically 3 main reasons why people are injured from explosions.

(i) Mechanical injury from the initial shockwave.

(ii) Injuries from flying debris/ objects caused by the explosion.

(iii) Physically colliding with objects owing to explosion shockwave.

Let us suppose a normal sized-human can sustain up to E_0 units of energy. The energy released from the explosion is E_e. The person is initially standing at a distance R from the centre of the explosion (Figure 4.5). Suppose all the energy from the explosion is uniformly released as a shock wave in the upper hemisphere from the centre. If the person facing the centre exposes an area of A_0 of his body, find the energy absorbed by the human. You may assume that locally the energy from the shockwave will be entirely absorbed by the object the wave comes into contact with. Hence find the condition for which he survives the blast.

(b) Now account for the flying debris. You can suppose that the debris is uniformly released from the centre of the explosion, each with a mass of dm and a velocity of u. Obtain an equation for the envelope of the explosion. This gives an "explosion" boundary which contains all the possible trajectories for fixed u (see Figure 4.6).

(c) Suppose this person is outside the envelope described in (b) and Figure 4.6. Model a person as a cube of side L. Suppose that the person takes up all the kinetic energy from the shockwave which the body has contact with (as in (a)). He flies backwards and hits a car with his back. Suppose $m < M$. If they are free to move only in a straight line, find the change in momentum of the man in the collision if the collision is elastic. Given that the contact surface area is A and the time of collision is Δt, find the average pressure on the human. Describe his possible injuries (Figure 4.7).

Solution: First a bit of disclaimer here. I have used a social context here to give you some ideas of how you can use the physics knowledge in a practical context. The diagrams are drawn in a cartoonish way but this is not meant as a joke - people have genuinely lost their lives and we, as physicists, are simply objectively analysing the problem to come up with suggestions to minimise loss of lives and property. As "leaders of the future", do remember that small decisions you make may have a huge impact on other people's lives.

Mechanical Shockwave We begin by discussing the mechanical shockwave. The main thing I would like you to learn is the concept of solid angle. The rigorous definition of solid angle is as follows.

Definition 4.1. *The infinitesimal* ***solid angle*** $d\Omega$ *from origin* $\mathcal{O}$ *is defined as*

$$d\Omega = \frac{\mathbf{dS} \cdot \mathbf{r}}{r^3} \tag{4.36}$$

where **dS** *is the infinitesimal surface element and* **r** *is the radial vector from the origin* $\mathcal{O}$ *(r is the magnitude of the vector) . The unit for solid angle is* ***steradians****.*

The solid angle for a full spherical surface enclosing the origin is 4π:

$$\Omega_{sphere} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta d\theta d\phi = 4\pi \tag{4.37}$$

So for a hemisphere the solid angle is 2π. Provided the man is sufficiently far away, he covers a solid angle of $\frac{A_0}{R^2}$. So the energy absorbed E_{ab} is:

$$E_{ab} = \frac{A_0}{2\pi R^2} E_e \tag{4.38}$$

The condition for survival is therefore:

$$E_0 > \frac{A_0}{2\pi R^2} E_e \tag{4.39}$$

Explosion Envelope Now we calculate the explosion envelope. We start with the equation of trajectory:

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta} x^2 \tag{4.11}$$

Rearranging gives:

$$\frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + y + \frac{gx^2}{2u^2} = 0 \tag{4.40}$$

where I have used $\sec^2\theta = 1 + \tan^2\theta$. This is a quadratic equation in $\tan\theta$ Since this requires real solutions, the determinant must be non-negative:

$$x^2 - 4(\frac{gx^2}{2u^2})(y + \frac{gx^2}{2u^2}) \geq 0 \tag{4.41}$$

The minimum velocity required to reach a certain angle is given when the above inequality becomes an equality:

$$1 = \frac{2g}{u^2}(y + \frac{gx^2}{2u^2}) \tag{4.42}$$

Solving for u^2 we have:

$$u^2 = gy \pm g\sqrt{x^2 + y^2} \tag{4.43}$$

To allow for a physical solution we must take the positive root. This gives:

$$u = \pm\sqrt{gy + g\sqrt{x^2 + y^2}} \tag{4.44}$$

Note that this means that the quadratic equation (Equation 4.40) has repeated roots. So this gives:

$$\tan\theta = \frac{u^2}{gx} \tag{4.45}$$

You can obtain this condition by setting $\frac{\partial y}{\partial \theta} = 0$. This is physically interpreted as to find the extrema angle to reach a certain height y given a velocity u. So substituting Equation 4.45 into the equation of trajectory (Equation 4.11) gives:

$$y = -\frac{gx^2}{2u^2} + \frac{u^2}{2g} \tag{4.46}$$

which is a parabolic envelope.

Blasting and Crashing Suppose all the energy is transferred to the kinetic energy of the man. We have from Equation 4.38:

$$\frac{1}{2}mu^2 = E_{ab} = \frac{A_0}{2\pi R^2}E_e \tag{4.47}$$

Recall from Chapter 1 that after an elastic collision with the second object initially stationary we have the final velocity:

$$v = \frac{m-M}{m+M}u \tag{4.48}$$

so the momentum change is

$$\Delta p = (1 - \frac{m-M}{m+M})mu \tag{4.49}$$

The pressure is hence:

$$P = \frac{2mM}{A\Delta t(m+M)}\sqrt{\frac{A_0 E_e}{m\pi R^2}} \tag{4.50}$$

The person is likely to die from internal rupture of organs from the impact with the car or mechanical injuries causing loss of blood. Solutions to the last part is open for interpretation.

Question 19: A particle is thrown with initial velocity u from the ground. The angle to horizontal at launch is θ. The particle path ends when it first hits the ground again. Find the angle of launch which

(a) Maximises the area covered by the particle.

(b) Maximises the path length travelled by the particle.

Solution: This question is not in HKPhO syllabus. It requires a sophisticated use of calculus.

Maximum Area The area covered by the motion is

$$A = \int y dx \tag{4.51}$$

Using the equation of trajectory (Equation 4.11) we have:

$$A = \int_0^R (x \tan\theta - \frac{gx^2}{2u^2\cos^2\theta})dx \tag{4.52}$$

Solving the integral gives:

$$A = \frac{2u^4 \sin^3\theta\cos\theta}{3g^2} \tag{4.53}$$

Maximising with respect to θ gives:

$$\frac{dA}{d\theta} = -\sin^4\theta + 3\sin^2\theta\cos^2\theta = 0 \tag{4.54}$$

This solves to give:

$$\sin\theta = \frac{\sqrt{3}}{2} \tag{4.55}$$

or

$$\theta = \frac{\pi}{3} \tag{4.56}$$

Maximum Length The path length travelled by the particle is given by

$$dl^2 = dx^2 + dy^2 \tag{4.57}$$

$$l = \int_0^{t_{max}} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \tag{4.58}$$

We know that $\frac{dx}{dt} = u\cos\theta$ and $\frac{dy}{dt} = u\sin\theta - gt$. Substitution gives:

$$l = u\cos\theta \int_0^{t_{max}} \sqrt{1 + (\frac{gt}{u\cos\theta} - \tan\theta)^2} dt \tag{4.59}$$

Use the substitution $\tan\phi = \frac{gt}{u\cos\theta} - \tan\theta$. The integration boundaries[1] are:

$$\begin{cases} \tan\phi|_{t=0} = -\tan\theta \\ \tan\phi|_{t=t_{max}} = \tan\theta \\ \sec\phi|_{t=0} = \sec\theta \\ \sec\phi|_{t=t_{max}} = \sec\theta \end{cases} \tag{4.60}$$

Using the standard integral

$$\int \sec^3\phi d\phi = \frac{1}{2}\sec\phi\tan\phi + \frac{1}{2}\log|\sec\phi + \tan\phi| \tag{4.61}$$

[1] You should check this on your own.

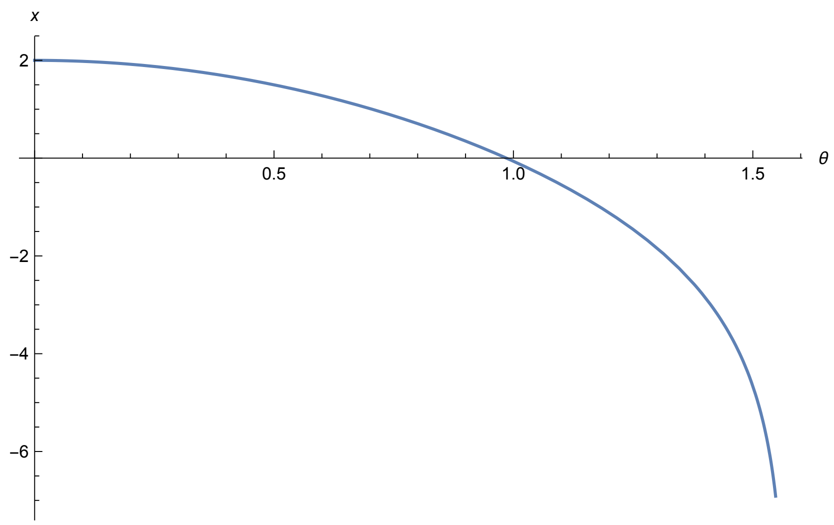
we hence have:

$$l = \frac{u^2 \sin\theta}{g} + \frac{u^2 \cos^2\theta}{2g} \log\left|\frac{1+\sin\theta}{1-\sin\theta}\right| \tag{4.62}$$

Setting $\frac{\partial l}{\partial \theta} = 0$ gives

$$2 - \sin\theta \log\left|\frac{1+\sin\theta}{1-\sin\theta}\right| = 0 \tag{4.63}$$

This is plotted in Figure 4.9 (shown again here).



You can see that the root occurs at about 1 radians. Using RootFinder this evaluates to:

$$\theta_{max} = 0.985515 = 56.4658^\circ \tag{4.64}$$

4.3 Take-home skills

The skills that you should now know are:

1. How to analyse problems regarding projectile motion.

2. Using the determinant argument to argue for real solutions.

4.4 Response Comments

General Comments

1. Q15: Most people who failed to finish the last part misunderstood my question. My apologies there...

2. Q16: A really straight-forward question. Most people got it right. Some failed to use converse of Pythagoras theorem!

3. Q17: Most who could not do it missed the condition. Some failed to use the determinant argument after figuring out the condition.

4. Q18: (a) is actually quite straightforward but apparently quite a few people got the surface area of a hemisphere wrong. (b) is a standard HKPhO-type question and most had trouble with using the determinant argument. (c) is straight-forward.

5. Q19: I am actually surprised someone finished (b) - going through the algebra is quite tough and well done!!

6. Overall: This is one of the best-performing exercise. Well done to you all. (Updated by Lucas on 21$^{\text{st}}$ August, 2020)

Chapter 5

Rod and Strings

In this chapter, we will talk about how to deal with questions with regards to rods and strings. Let us first review a few definitions surrounding these objects.

Definition 5.1. *An **ideal string** is*

1. *Inextensible - it cannot stretch.*
2. *Massless - it has no mass.*

Definition 5.2. *An **ideal massless rod***

1. *Has no mass.*
2. *Is non-extensible.*

Definition 5.3. *An **ideal spring***

1. *Acts in 1D.*
2. *Is massless.*
3. *Obeys Hooke's Law for small changes in length x.*

Note that if an object is massless, there must not be any net force or torque on the object. Otherwise this would result in an infinite acceleration, something not allowed physically.

5.1 Questions

Question 20: Consider two spheres of mass $2m$ and m connected by an ideal rigid rod initially lying at rest on a smooth horizontal plane.

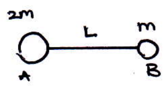


Figure 5.1: Ideal rigid rod with end masses of $2m$ and m.

(a) Perpendicular Impulse. (b) Impulse at an angle θ.

Figure 5.2: Impulse given on the system at different directions.

(a) Let us suppose ball B is given on initial impulse of J perpendicular to the rod (Figure 5.2a). Find the tension in the rod. Also find the period of revolution of the system.

(b) Repeat the calculation when the impulse is given at an angle θ to the axis of the system (Figure 5.2b).

Question 21: A ball of mass m is attached to the ceiling by an ideal string of length L. The system is initially at rest. The angle that the string makes with the vertical is θ (Figure 5.3).

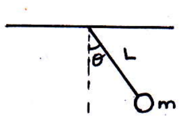


Figure 5.3: Pendulum system with ball of mass m and string of length L.

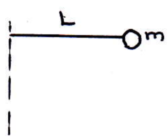
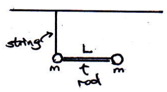


Figure 5.4: Configuration in (b).

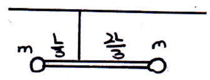
(a) Find the acceleration and tension just after the system is released from rest.

(b) Let us suppose that initially the angle $\theta = \frac{\pi}{2}$ (Figure 5.4). If the ball is released from rest, find the tension and acceleration in terms of θ.

Question 22: In these following systems, two point masses of mass m are connected to the ends of an ideal rigid rod. The systems are released from rest. Find the tension in the rod just after release. (Also find the accelerations of the balls just after release.)

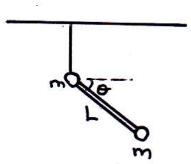


(a) Set up (a).

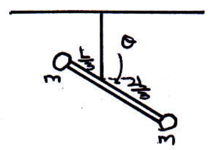


(b) Set up (b).

Figure 5.5: Set ups of rigid rod with end masses m.



(a) Set up (c).



(b) Set up (d).

Figure 5.6: Set ups of rigid rod with end masses m at an angle θ to the horizontal.

Question 23: Consider a rod with its ends connected to m_1 and m_2 respectively initially at rest on the surfaces of a smooth corner as illustrated in Figure 5.7. A box is put next to the rod and the lower ball and the box are not initially in contact with each other.

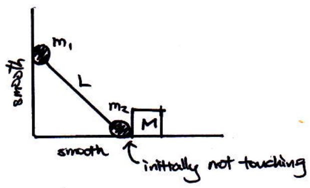


Figure 5.7: Two balls of mass m_1 and m_2 connected to the ends of the rod sliding down a smooth corner. The box and the lower mass is not initially in contact with each other.

(a) Assume (only in this part) that $m_1 = m_2 = m$. Find the initial accelerations of the masses if the initial angle between the rod and the vertical wall is θ_0.

(b) Now the system is released from rest. Find the angle θ that the rod makes with the vertical wall when the rod detaches from the wall. Assume that $m_1 = m_2 + M$.

(c) After the detachment, find the vertical velocity of the upper ball when it is just about to hit the floor.

Question 24: Consider three identical balls on a horizontal plane. The two outer balls are connected to the middle by two strings of length l.

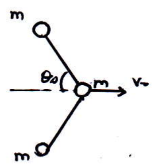


Figure 5.8: Three identical balls of mass m with two strings of length l.

Initially the two outer balls are placed such that the strings are not slackened and they make an angle θ_0 to the horizontal. The middle ball is given an initial impulse of I such that the centre ball moves at a velocity of v_0.

(a) Find the relationship between I and v_0. Hence find the initial velocities of the outer balls.

(b) When the outer balls first collides with each other, find the velocity of one of these balls.

(c) Suppose a particle is moving in plane polars. Derive the following acceleration expression:

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{5.1}$$

Using this result, or otherwise, derive the equation of motion of the system.

5.2 Solutions and Commentary

Question 20: Consider two spheres of mass $2m$ and m connected by an ideal rigid rod initially lying at rest on a smooth horizontal plane. Suppose ball B is given on initial impulse of J perpendicular to the rod (Figure 5.2a). Find the tension in the rod. Also find the period of revolution of the system. Repeat the calculation when the impulse is given at an angle θ to the axis of the system (Figure 5.2b).
Solution: These problems are all easily dealt with using techniques from analysing rigid body dynamics. However, it is also possible to analyse the problem in terms of Newtonian laws. Here I present two methods to approach the same problem. The Newtonian solution is kindly provided by Albus Poon whom I am indebted to.

Newtonian Laws

Rigid Body Dynamics We first calculate the moment of inertia about the centre of mass of the system.

$$I = m(\frac{2L}{3})^2 + 2m(\frac{L}{3})^2 = \frac{2}{3}mL^2 \tag{5.2}$$

The angular speed can be calculated from the gain in angular momentum.

$$L = J\frac{2L}{3} = I\omega \tag{5.3}$$

which solves to give:

$$\omega = \frac{2LJ}{3I} \tag{5.4}$$

Substituting the moment of inertia I gives,

$$\omega = \frac{2LJ}{2mL^2} = \frac{J}{mL} \tag{5.5}$$

The tension can be calculated from the centripetal acceleration of one of the balls. Here consider ball B.

$$T = (\frac{2L}{3})(\frac{J}{mL})^2 = \frac{2J^2}{3mL} \tag{5.6}$$

For the second part simply replace J by $J\sin\theta$.

Question 21: A ball of mass m is attached to the ceiling by an ideal string of length L. The system is initially at rest. The angle that the string makes with the vertical is θ (Figure 5.3). Find the acceleration and tension just after the system is released from rest. Now suppose that initially the angle $\theta = \frac{\pi}{2}$ (Figure 5.4). If the ball is released from rest, find the tension and acceleration in terms of θ.
Solution: The first part to the problem is straight-forward. The free-body diagram of the system is sketched in Figure 5.9.

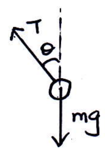


Figure 5.9: The free-body diagram for the string-mass system.

There is no acceleration parallel of the massless string. This leaves the tangential acceleration of the ball, therefore,

$$a = g\sin\theta \tag{5.7}$$

The tension along the string is trivially obtained.

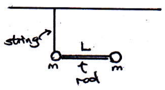
$$T = mg\cos\theta \tag{5.8}$$

Using the same argument as above, as the system evolves from rest at $\theta = \frac{\pi}{2}$, the tension and tangential acceleration is also

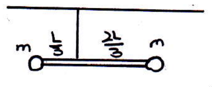
$$a_{\tan} = -g\sin\theta \tag{5.9}$$

$$T = mg\cos\theta \tag{5.10}$$

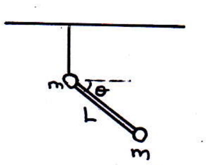
Question 22: In these following systems, two point masses of mass m are connected to the ends of an ideal rigid rod. The systems are released from rest. Find the tension in the rod just after release. (Also find the accelerations of the balls just after release.)



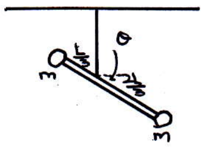
(a) Set-up 1.



(b) Set-up 2.



(a) Set-up 3.



(b) Set-up 4.

Solution: I will only calculate the tension in the string using rigid body dynamics. The acceleration of the balls can be easily obtained by substituting the (calculated) value of the tension back to the equations of motion to find the accelerations $\ddot{x}$ of the centre of mass and the angular acceleration $\ddot{\theta}$. The resulting acceleration is found by

$$a_{\text{res}} = \ddot{x} + r\ddot{\theta} \tag{5.11}$$

where r is the distance of the object from the centre of mass.

Set-up 1 Consider vertical forces (Figure 5.12). This gives

$$2m\ddot{y} = 2mg - T \tag{5.12}$$

The rotational version of the Newton's Second Law gives:

$$I\ddot{\theta} = T\frac{L}{2} \tag{5.13}$$

where $I = \frac{1}{2}mL^2$ is the moment of inertia of the rod about its centre of mass. By noting that there is no vertical forces at the leftmost ball (as it is connected to a massless string), we can write:

$$\ddot{y} = \frac{L}{2}\ddot{\theta} \tag{5.14}$$

Substituting the expressions for $\ddot{x}$ and $\ddot{\theta}$ gives

$$\frac{2mg - T}{2m} = \frac{T}{2m} \tag{5.15}$$

This reduces to

$$T = mg \tag{5.16}$$

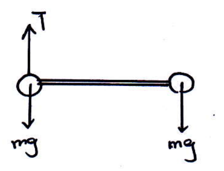


Figure 5.12: Free-body diagram of Set-up 1.

Set-up 2 The vertical NII[1] are the same as found in Equation 5.12. The rotational NII gives (Figure 5.13):

$$T\frac{L}{6} = I\ddot{\theta} \tag{5.17}$$

Using the moment of inertia $I = \frac{1}{2}mL^2$,

$$\ddot{\theta} = \frac{T}{3mL} \tag{5.18}$$

There is no net acceleration at the attachment point with the rod and the string. This condition is mathematically expressed as:

$$\ddot{x}_{vert} = \frac{L}{6}\ddot{\theta} = \frac{T}{18m} \tag{5.19}$$

Hence after substitution we have

$$T = \frac{9}{5}mg \tag{5.20}$$

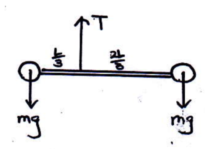


Figure 5.13: Free-body diagram of Set-up 2.

Set-up 3 Consider the free-body diagram of the set-up. The vertical NII is the same as before (Equation 5.12). The rotational NII gives (Figure 5.14):

$$\frac{L}{2}T\cos\theta = I\ddot{\theta} = \frac{1}{2}mL^2\ddot{\theta} \tag{5.21}$$

[1]NII means Newton's Second Law of Motion.

This reduces to

$$\ddot{\theta} = \frac{T\cos\theta}{mL} \tag{5.22}$$

Note that there is no vertical acceleration at the point of contact of the string and the rod. This gives the condition:

$$\ddot{y} = \frac{L}{2}\ddot{\theta}\cos\theta \tag{5.23}$$

This gives

$$\frac{T\cos^2\theta}{2m} = \frac{2mg - T}{2m} \tag{5.24}$$

Solving for T gives

$$T = \frac{2mg}{1+\cos^2\theta} \tag{5.25}$$

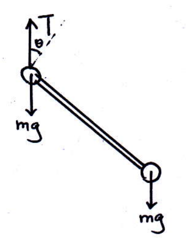


Figure 5.14: Free-body diagram of Set-up 3.

Set-up 4 Consider the free-body diagram of the set-up (Figure 5.15). The vertical NII is the same as before (Equation 5.12, again!). The rotational NII gives

$$\frac{L}{6}T\cos\theta = I\ddot{\theta} \tag{5.26}$$

This gives

$$\ddot{\theta} = \frac{T\cos\theta}{3mL} \tag{5.27}$$

There is no vertical acceleration at the point of contact of the string and the rod. This is mathematically expressed as the following

$$\ddot{y} = \frac{L}{6}\ddot{\theta}\cos\theta \tag{5.28}$$

Substitution for $\ddot{y}$ and $\ddot{\theta}$ gives

$$\frac{2mg - T}{2m} = \frac{T\cos^2\theta}{18m} \tag{5.29}$$

Solving for T gives

$$T = \frac{18mg}{9+\cos^2\theta} \tag{5.30}$$

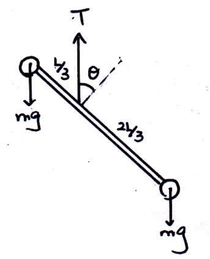


Figure 5.15: Free-body diagram of Set-up 4.

Question 23: Consider a rod with its ends connected to m_1 and m_2 respectively initially at rest on the surfaces of a smooth corner as illustrated in Figure 5.7. A box is put next to the rod and the lower ball and the box are not initially in contact with each other.

(a) Assume (only in this part) that $m_1 = m_2 = m$. Find the initial accelerations of the masses if the initial angle between the rod and the vertical wall is θ_0.

(b) Now the system is released from rest. Find the angle θ that the rod makes with the vertical wall when the rod detaches from the wall. Assume that $m_1 = m_2 + M$.

(c) After the detachment, find the vertical velocity of the upper ball when it is just about to hit the floor.

Solution: This problem is extremely similar to ladder question in Ch2. This is split into the following parts.

Initial Accelerations The initial condition is solved in a similar method to the questions above. First consider the free-body diagram of the rod.

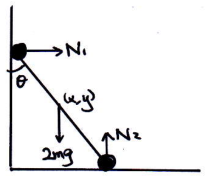


Figure 5.16: Free-body diagram of the rod.

Newton's Second Law for horizontal, vertical and rotational directions give the following set of equations:

$$\begin{cases} m\ddot{x} = N_1 \\ m\ddot{y} = N_2 - 2mg \\ I\ddot{\theta} = (N_2 \sin\theta - N_1 \cos\theta)\,\frac{L}{2} \end{cases} \tag{5.31}$$

where (x, y) is the coordinates of the centre of mass. Eliminating N_1 and N_2 gives

$$I\ddot{\theta} = (2mg + m\ddot{y})\sin\theta - m\ddot{x}\cos\theta \tag{5.32}$$

Now we need to find expressions for $\ddot{x}$ and $\ddot{y}$. Simple geometry gives:

$$(x, y) = \left(\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\right) \tag{5.33}$$

Differentiating twice gives:

$$(\dot{x}, \dot{y}) = \left(-\frac{L}{2}\sin\theta\dot{\theta}, \frac{L}{2}\cos\theta\dot{\theta}\right) \tag{5.34}$$

$$(\ddot{x}, \ddot{y}) = \left(-\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta}, -\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{L}{2}\cos\theta\ddot{\theta}\right) \tag{5.35}$$

Initially $\dot{\theta} = 0$. This solves to give:

$$(\ddot{x}_0, \ddot{y}_0) = \left(-\frac{L}{2}\sin\theta\ddot{\theta}, \frac{L}{2}\cos\theta\ddot{\theta}\right) \tag{5.36}$$

Hence, substituting this into Equation 5.32 gives:

$$\ddot{\theta}_0 = \frac{2g\sin\theta}{L} \tag{5.37}$$

This solves to give the initial accelerations:

$$\ddot{x}_0 = -g\sin^2\theta \tag{5.38}$$

$$\ddot{y}_0 = g\cos\theta\sin\theta \tag{5.39}$$

To obtain the accelerations of the balls, use Equation 5.11 again. This is left as an exercise.

Detachment Angle Conservation of energy gives

$$mgL(\cos\theta_0 - \cos\theta) = \frac{1}{2}m_1v_y^2 + \frac{1}{2}m_2v_x^2 + \frac{1}{2}Mv_x^2 \tag{5.40}$$

This reduces to

$$2gL(\cos\theta_0 - \cos\theta) = v_x^2 + v_y^2 \tag{5.41}$$

Since the rod is a rigid body, all quantities along the direction of the rod must be conserved. So velocities parallel to the direction of the rod is the same (Figure 5.17), giving

$$v_y\cos\theta = v_x\sin\theta \tag{5.42}$$

Substitution into Equation 5.41 gives

$$v_x^2 = 2gL\cos^2\theta(\cos\theta_0 - \cos\theta) \tag{5.43}$$

Maximise to give

$$\cos\theta = \frac{2}{3}\cos\theta_0 \tag{5.44}$$

This is the angle of detachment of the rod.

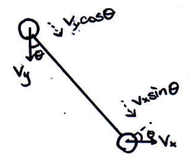


Figure 5.17: Velocities along the rod.

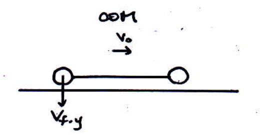


Figure 5.18: The system when the rod is about to hit the floor.

Velocity of upper ball when it hits the floor At the point of detachment the velocity of the ball is

$$v_x^2 = 2gL\left(\frac{2}{3}\cos\theta_0\right)^2\left(\frac{1}{3}\cos\theta_0\right) \tag{5.45}$$

This gives

$$v_x^2 = \frac{8gL}{27}\cos^3\theta_0 \tag{5.46}$$

After detachment from the wall the horizontal momentum is conserved. The velocity of the centre of mass of the rod when the system hits the ground is

$$v_0 = \frac{m_2 v_x}{m_1 + m_2} \tag{5.47}$$

With the quantities defined in Figure 5.18, we now use conservation of energy:

$$\frac{1}{2}(m_1+m_2)v_0^2 + \frac{1}{2}m_1 v_{f,y}^2 = \frac{1}{2}m_2 v_x^2 + \frac{1}{2}m_1 v_y^2 \tag{5.48}$$

This gives

$$v_{f,y}^2 = \frac{m_2}{m_1+m_2}v_x^2 + v_y^2 \tag{5.49}$$

Note that at detachment,

$$v_y = v_x \tan\theta \tag{5.50}$$

where

$$\tan\theta = \frac{\sqrt{1-\frac{4}{9}\cos^2\theta_0}}{\frac{2}{3}\cos\theta_0} \tag{5.51}$$

Therefore we have

$$v_{f,y}^2 = \left[\frac{8gL}{27}\cos^3\theta_0\right]\left[\frac{m_2}{m_1+m_2} + \frac{1-\frac{4}{9}\cos^2\theta_0}{\frac{4}{9}\cos^2\theta_0}\right] \tag{5.52}$$

Question 24: Consider three identical balls on a horizontal plane. The two outer balls are connected to the middle by two strings of length l. Initially the two outer balls are placed such that the strings are not slackened and they make an angle θ_0 to the horizontal. The middle ball is given an initial impulse of I such that the centre ball moves at a velocity of u_0.

(a) Find the relationship between I and u_0. Hence find the initial velocities of the outer balls.

(b) When the outer balls first collides with each other, find the velocity of one of these balls.

(c) Derive the equation of motion of the system.

Solution: This question is similarly adapted from an SYSSPhO Question.

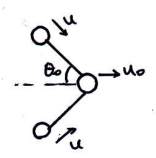


Figure 5.19: The quantities of the system after the initial impulse is given.

Relationship between I and u_0 Since the rod is a rigid body, quantities along the rod are conserved. Therefore, we have

$$u = u_0 \cos\theta_0 \tag{5.53}$$

where u is the initial velocity of the outer balls. The horizontal impulse is

$$I = mu_0 + 2mu\cos\theta_0 \tag{5.54}$$

Hence

$$I = mu_0 + 2mu_0\cos^2\theta_0 \tag{5.55}$$

This is the relation between I and u_0. Then

$$u_0 = \frac{I}{m(1+2\cos^2\theta_0)} \tag{5.56}$$

Therefore,

$$u = \frac{I\cos\theta_0}{m(1+2\cos^2\theta_0)} \tag{5.57}$$

Velocity of outer balls at collision Note that horizontal momentum is conserved. Therefore, at the point where the two balls collide with each other, the centre ball has a velocity

$$v_0 = \frac{I}{3m} \tag{5.58}$$

Figure 5.20: The quantities of the system when the outer balls collide with each other. All the balls have the same horizontal velocity.

Conservation of energy gives

$$\frac{1}{2}mu_0^2 + mu^2 = \frac{3}{2}mv_0^2 + mv^2 \tag{5.59}$$

This reduces to

$$u_0^2 + 2u^2 = 3v_0^2 + 2v^2 \tag{5.60}$$

Substituting Equation 5.53 and rearranging gives

$$v^2 = \frac{1}{2}\left[u_0^2(1+\cos^2\theta_0) - 3v_0^2\right] \tag{5.61}$$

Substituting expressions for u_0 and v_0 (Equations 5.56 and 5.58) gives

$$v^2 = \frac{1}{2}\left[\frac{I^2}{m^2(1+2\cos^2\theta_0)^2} - \frac{I^2}{3m^2}\right] \tag{5.62}$$

Hence

$$v = \frac{I}{m(1+2\cos^2\theta_0)}\sqrt{\frac{1-2\cos^2\theta_0-2\cos^4\theta_0}{3}} \tag{5.63}$$

Equation of Motion The equation of motion is easily derived by considering the evolution of the system in the non-inertial frame of the middle ball. By considering the free-body diagram of the middle ball, NII along the horizontal direction gives

$$m\ddot{x} = 2T\cos\theta \tag{5.64}$$

The acceleration of the centre ball is hence

$$\ddot{x} = \frac{2T\cos\theta}{m} \tag{5.65}$$

So in the non-inertial frame of the middle ball, the free-body diagram is

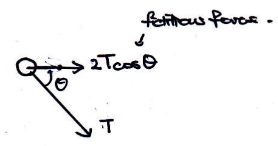


Figure 5.21: Free-body diagram of the outer (upper) ball in the non-inertial frame of the middle ball.

Recall that the acceleration of a particle in plane polars is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{5.1}$$

Note that $\ddot{r} = \dot{r} = 0$ since the string is stretched at all times. Then NII in the radial and tangential directions give

$$T(1 + 2\cos^2\theta) = mr\dot{\theta}^2 \tag{5.66}$$

$$2T\cos\theta\sin\theta = mr\ddot{\theta} \tag{5.67}$$

respectively. Eliminating T gives

$$\ddot{\theta} = \frac{2\cos\theta\sin\theta}{1 + 2\cos^2\theta}\dot{\theta}^2 \tag{5.68}$$

This is the equation of motion of the system. Interesting, the angular evolution of the system is shown in the following graphs.

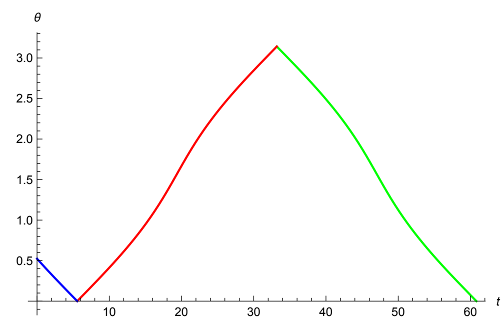


Figure 5.22: A plot of the θ evolution with time. The horizontal axis is time and the vertical axis is θ. Here the initial conditions are $\theta_0 = \frac{\pi}{6}$ and $\dot{\theta}_0 = -0.1$. The different colours show the separately numerically integrated solutions, with the hard-sphere collision implemented manually at the boundaries (see the Appendix for further explanation).

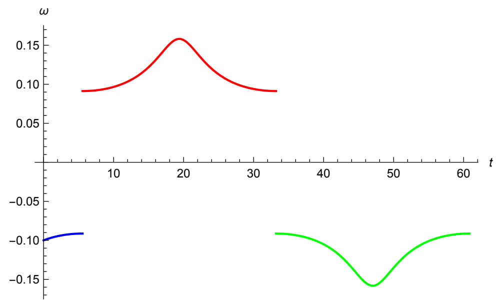


Figure 5.23: A plot of the $\dot{\theta}$ evolution with time. The horizontal axis is time and the vertical axis is $\dot{\theta} = \omega$. Here the initial conditions are $\theta_0 = \frac{\pi}{6}$ and $\dot{\theta}_0 = -0.1$. The different colours show the separately numerically integrated solutions, with the hard-sphere collision implemented manually at the boundaries (see the Appendix for further explanation).

This shows that the outer-balls travel the fastest when all the three balls are travelling in the same direction. This matches up with our intuition! The same result can be done by performing a Lagrangian analysis on the system. This method, together with a guide on plotting the figures in shown in the Appendix.

5.3 Take-home skills

The skills that you should now know are:

1. Conditions for rigid bodies - there is no internal relative motion.

2. Practising elementary physics techniques on rigid body problems.

Chapter 6

Detachment

This chapter is on problems relating to the detachment of an object with something. This is a really common condition for objects in classical dynamics, typically signalling the need for change of coordinates due to the different constraints provided by the problem. We need not, however, go that deep in the analysis route. All you have to remember is that in the event of detachment any contact forces would simply vanish, and applying conservation laws and force analysis would suffice at that point.

6.1 Questions

Question 25: In this question we will discuss the problem of balls and spherical surfaces.

(a) Consider a ball of mass m (which you can consider as dimensionless) put on top of a hemisphere of radius R. The hemisphere is fixed on the ground. The ball is given a small push push from rest. Find the angle for which it detaches from the hemisphere.

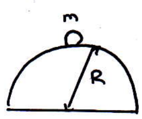


Figure 6.1: A ball resting on top of a hemisphere of radius R. The ball is given a small horizontal perturbation.

(b) Now consider the following set-up: a "slide" made up by two quarter-circles of radius R, one convex and one concave. A ball is initially put at a height H above the ground on the track. It is released from rest. Find:

(i) The velocity of the ball when it reaches the middle point C.

(ii) The angle θ from the vertical of C when the ball detaches from the surface.

(iii) The horizontal displacement from the original position when the ball lands on the floor.

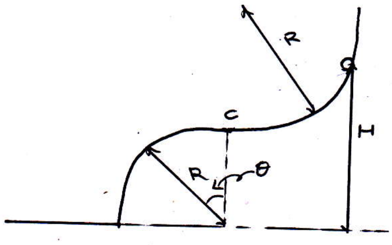


Figure 6.2: The track made up of circular parts. The ball is put on the track at H from the ground.

Question 26: A ball is connected to the end of a massless rigid rod. It is given a push such that the ball has an initial velocity of v_0.

Figure 6.3: Pendulum of length L and bob mass of m given a initial velocity v_0 to the right.

Find the minimum velocity of the system to complete a full cycle of motion.

Question 27: Consider a ball connected to the end of an ideal string of length L. Tha ball is released from rest when the string is horizontal.

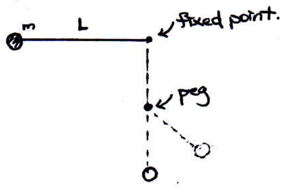


Figure 6.4: Pendulum with peg at distance $\frac{L}{2}$ down the vertical of the connection point.

There is a peg $\frac{L}{2}$ down point P where the string is nailed to the wall.

(a) Find the velocity of the ball when the string is vertical.

(b) Find the angle θ which the string first becomes slackened. The angle is measured from the vertical from point P (so like a clock).

(c) Describe the motion of the ball after the string is just slackened.

(d) Remove the peg. What happens if the string is replaced by a rod and the ball is given an initial downward velocity?

Question 28: A box is placed on a spring balance (Figure 6.5). The spring is compressed by x_0. Suppose the spring is further pushed down such that it is now x_1 from the equilibrium of the system. Find the extension length of the spring when the box detaches from the stage of the spring balance system.

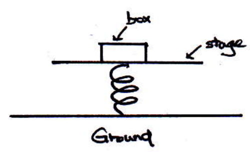


Figure 6.5: A box on top of a spring balance.

6.2 Solutions and Commentary

Question 25: In this question we will discuss the problem of balls and spherical surfaces.

(a) Consider a ball of mass m (which you can consider as dimensionless) put on top of a hemisphere of radius R. The hemisphere is fixed on the ground. The ball is given a small push push from rest. Find the angle for which it detaches from the hemisphere.

(b) Now consider the following set-up: a “slide” made up by two quarter-circles of radius R, one convex and one concave (Figure 6.2). A ball is initially put at a height H above the ground on the track. It is released from rest. Find the velocity of the ball and the angle from vertical θ when the ball detaches from the surface. Hence find the horizontal displacement from the original position when the ball lands on the floor.

Solution: This is a standard question considering detachment of objects.

Ball and Hemisphere First draw a free-body diagram of the ball.

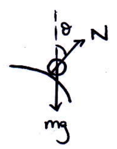


Figure 6.6: Free-body diagram of the ball on the hemisphere.

Using conservation of energy, we have that

$$mgR(1 - \cos\theta) = \frac{1}{2}mv^2 \tag{6.1}$$

Solving for v^2 gives

$$v^2 = 2gR(1 - \cos\theta) \tag{6.2}$$

Now the centripetal force acting on the ball gives

$$m\frac{v^2}{R} = mg\cos\theta - N \tag{6.3}$$

When ball detaches from the hemisphere, the ball is no longer “pressing” on the hemisphere. Therefore the normal force is vanishes, and

$$v^2 = gR\cos\theta \tag{6.4}$$

Equating v^2 in Equations 6.4 and 6.8 gives

$$gR\cos\theta = 2gR - 2gR\cos\theta \tag{6.5}$$

This solves to give

$$\cos\theta = \frac{2}{3} \tag{6.6}$$

The "slide" Suppose the velocity of the ball at the centre of the slide (Point C) is v_0. Conserving energy gives

$$mg(H - R) = \frac{1}{2}mv_0^2 \tag{6.7}$$

This solves to give

$$v_0^2 = 2g(H - R) \tag{6.8}$$

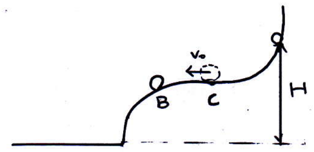


Figure 6.7: Diagram of the slides with the relevant quantities labelled.

Now suppose the ball detaches at point B. The free-body diagram is exactly the same (the mirror image of) as that illustrated in the first part of the question. Conserving energy with respect to points B and C gives

$$v^2 - v_0^2 = 2gR(1 - \cos\theta) \tag{6.9}$$

Substituting Equation 6.8 and rearranging gives

$$v^2 = 2gH - 2gR\cos\theta \tag{6.10}$$

NII in the radial direction is

$$m\frac{v^2}{R} = mg\cos\theta - N \tag{6.11}$$

Now setting $N = 0$ at the point of detachment gives

$$v^2 = gR\cos\theta \tag{6.12}$$

Hence using Equation 6.10 gives

$$3gR\cos\theta = 2gH \tag{6.13}$$

giving

$$\cos\theta = \frac{2H}{3R} \tag{6.14}$$

Note that here there is an additional condition that

$$H \leq \frac{3R}{2} \tag{6.15}$$

This reflects the fact that if H exceeds this value, the ball would simply detach at point C of the slide and fly straight off[1].

To find the horizontal displacement from the initial position we use the equation of trajectory

$$-R\cos\theta = -x\tan\theta + \frac{gx^2}{2v^2\cos^2\theta} \tag{6.16}$$

where v is the velocity of the ball at Point B of the slide. Using Equation 6.10 and solving the quadratic equation gives:

$$x = R\cos^3\theta(-\tan\theta + \sqrt{2\sec^2\theta + \tan^2\theta}) \tag{6.17}$$

where we have taken the positive root (the negative root is unphysical). Here the angle θ is given by Equation 6.14. To account for the total displacement, however, one must also consider the horizontal displacement from the original position to Point B. This evaluates to:

$$x_{total} = x + R\sin\theta + (2R - H)\tan\theta \tag{6.18}$$

where the second term comes from the horizontal displacement between B and C and the last term is from considering the horizontal displacement between point C and the initial release point.

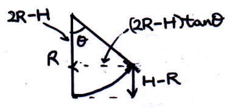


Figure 6.8: Geometry for finding the third term of the total horizontal displacement.

[1]This condition can be obtained by considering the NII and centripetal acceleration at point B of the slide.

Question 26: A ball is connected to the end of a massless rigid rod. It is given a push such that the ball has an initial velocity of v_0. Find the minimum velocity of the system to complete a full cycle of motion.

Solution: To find the required condition, note that we only need to check the point when the ball reaches the highest point. This is when the ball has the smallest velocity. Conserving energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + 2mgL \tag{6.19}$$

This gives

$$v_0^2 = v^2 + 4gL \tag{6.20}$$

If the ball is given the minimum velocity, there is no tension in the string at the highest point so

$$mg = m\frac{v^2}{L} \tag{6.21}$$

so

$$v^2 = gL \tag{6.22}$$

Therefore, using Equation 6.20 this gives

$$v_0^2 = 5gL \tag{6.23}$$

This is the minimum velocity as required.

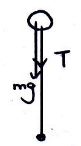


Figure 6.9: Free-body diagram of the ball connected to the rod when it is at the uppermost position.

Question 27: Consider a ball connected to the end of an ideal string of length L. Tha ball is released from rest when the string is horizontal. There is a peg $\frac{L}{2}$ down point P where the string is nailed to the wall. Find the angle θ which the string first becomes slackened (the angle is measured from the vertical from point P (like a clock)). Describe the motion of the ball after the string is just slackened. What happens if the string is replaced by a rod and the ball is given an initial downward velocity (assume there is no peg in this situation)?

Solution: A standard question on strings and rods again. The first part on finding the velocity of the ball v_0 when the string is vertical is trivial - conservation of energy gives

$$\frac{1}{2}mv_0^2 = mgL \tag{6.24}$$

giving

$$v_0 = \sqrt{2gL} \tag{6.25}$$

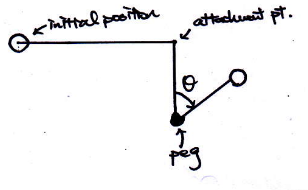


Figure 6.10: The diagram when the string becomes slacked. This shows the definition of the angle θ.

Now consider the point when the string becomes slackened. Conservation of energy gives

$$\frac{1}{2}mv^2 = \frac{L}{2}mg(1 - \cos\theta) \tag{6.26}$$

Solving for v^2 gives

$$v^2 = gL(1 - \cos\theta) \tag{6.27}$$

At the detachment point, the tension in the string is zero. Therefore, NII in the radial direction gives

$$mg\cos\theta = m\frac{v^2}{L} \tag{6.28}$$

$$v^2 = \frac{1}{2}gL\cos\theta \tag{6.29}$$

Using Equation 6.27 and 6.29 gives

$$\cos\theta = \frac{2}{3} \tag{6.30}$$

What happens after the string is detached? Similar to the motion in the last question, the ball is only under the influence of gravity. Therefore the ball undergoes projectile motion. What happens next depends on whether the ball passes the peg from above or not - if it does, then the ball undergoes an anharmonic oscillation. If it doesn't, then the ball travels to the other side and the same thing happens again (or the string breaks...).

Replacing the string by a rod A rod can provide a pulling or pushing force. In this case, the ball is constrained to move in a circular motion. Therefore the projectile motion described above will not happen (as a rod would not become "slackened") and the rod simply oscillates anharmonically between the extreme points of the motion.

Question 28: A box is placed on a spring balance (Figure 6.5). The spring is compressed by x_0. Suppose the spring is further pushed down such that it is now x_1 from the new equilibrium of the system. Find the extension length of the spring when the box detaches from the stage of the spring balance system.
Solution: The new equilibrium position of the system is found by considering forces acting on the box.

$$mg = kx_0 \tag{6.31}$$

giving

$$x_0 = \frac{mg}{k} \tag{6.32}$$

The equation of motion is

$$m\ddot{x} = -kx - mg \tag{6.33}$$

The solution to the equation is

$$x = A\cos(\omega_0 t) - x_0 \tag{6.34}$$

The initial condition gives that

$$x(0) = -x_0 - x_1 = A - x_0 \tag{6.35}$$

This gives

$$A = -x_1 \tag{6.36}$$

Hence this gives the acceleration as

$$\ddot{x} = \omega_0^2 x_1 \cos(\omega t) \tag{6.37}$$

The box detaches from the platform when the acceleration (downward) of the motion is equal to the gravitational acceleration. Mathematically this is expressed as

$$\ddot{x} = -g \tag{6.38}$$

Hence using Equation 6.37 gives

$$\omega_0^2 x_1 \cos(\omega t) = -g \tag{6.39}$$

$$t = \frac{1}{\omega_0}\left[\cos^{-1}\left(\frac{g}{\omega_0^2 x_1}\right) + \frac{\pi}{2}\right] \tag{6.40}$$

6.3 Take-home Skills

1. Usage of conservation laws.

2. Dealing with detachment problems.

Chapter 7

Gravitation

Newtonian Gravitation is one of the oldest yet most-used theories in physics. It is developed by Sir Isaac Newton and here we make a short summary of the theory. The proposition that Newton suggested was that the gravitational force is a central field force which has an inverse-square dependence on the radial separation from the source. Here we make some quick definitions and postulates.

Definition 7.1. *A **central force** is a force that only depends on the radial separation from the source to the object. It acts radially out-of or towards the source. Mathematically we write*

$$\mathbf{F} = f(r)\mathbf{e_r} \tag{7.1}$$

where $f(r)$ is a function that depends only on the radial distance from the object to the source r.

Postulate 7.1. *Newton postulated the gravitation force to be of the following form*

$$\mathbf{F} = -\frac{GMm_G}{r^2}\mathbf{e_r} \tag{7.2}$$

where G is the gravitational constant and M is the mass of the source. m_G is the gravitational mass of the object, which is different from the inertial mass m_I. r is the radial distance between the two masses (the source and the object).

Postulate 7.2. *There is no **a priori** reason why the gravitational mass and the inertial mass is equal. It is postulated that*

$$\frac{m_G}{m_I} = const \tag{7.3}$$

Experimentally, they are shown to be approximately equal to each other. It is therefore typical to set the constant to be 1. We therefore drop the subscripts for convenience.

Under a pure central force (field), an object undergoes orbital motion around a massive object (which acts as a source for this field). For gravitational forces, these orbits are conic sections (circular, elliptical, parabolic, hyperbolic). Here we only deal with *closed* orbits, which do not extend to infinity[1] Since the systems are normally isolated and the force is acting in the radial direction (which exerts no torque on the orbiting object), the energy and angular momentum of the motion are both conserved. This has the following consequences.

Postulate 7.3. *The entire motion is contained in a plane (a 3D subspace). This follows from the conservation of momentum.*

Definition 7.2. *The gravitational potential energy $V(r)$ is defined as*

$$V(r) = -\frac{GMm}{r} \tag{7.4}$$

This expression can be found by integrating the force law with respect to radial distance to find the work done to take the object to infinity. The integration constant is set by $V(\infty) = 0$.

We can write the total energy as the following

$$E = \frac{1}{2}mv^2 - \frac{A}{r} \tag{7.5}$$

where $A = GMm$. Since the motion is constrained in a plane, we can use plane polars:

$$\mathbf{v} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta \tag{7.6}$$

Therefore the total energy can be written as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{A}{r} \tag{7.7}$$

The angular momentum J in the system is

$$J = mr^2\dot{\theta} \tag{7.8}$$

Therefore Equation 7.7 becomes

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J}{2mr^2} - \frac{A}{r} \tag{7.9}$$

Therefore we can identify the effective potential energy as follows:

$$U_{\text{eff}} = \frac{J}{2mr^2} - \frac{A}{r} \tag{7.10}$$

Also the total energy in an orbit is (stated without proof)

$$E = -\frac{GMm}{2a} \tag{7.11}$$

[1]There is a question on open orbits which you can practice on.

where a is the semi-major axis of the elliptical closed orbit.

There are two more laws/theorem that I would like to state. First are the Kepler's Laws of planetary motion, which are empirical laws derived from experiments.

Law 7.1. *(Kepler's Laws of Planetary Motion) These laws state the following:*

(I) The orbit of a planet round the sun is an ellipse with the Sun at one focus.

(II) The line joining the Sun to a particular planet (the radius vector) sweeps out equal areas in equal time.

(III) The square of the period of the orbit of a planet around the Sun is proportional to the cube of its semi-major axis.

These empirical laws apply to the Solar system.

The second is the Shell Theorem, proved by Sir Isaac Newton in *Mathematica.*

Theorem 7.1. *(Shell Theorem) The shell theorem gives gravitational simplification to spherical bodies. There are two parts to the theorem:*

*(I) Outside a **uniform** spherical shell, the gravitational mass can be considered as concentrated at the centre of the sphere.*

*(II) Inside a **uniform** spherical shell, the net gravitational contribution from the shell is zero.*

We will practice all these techniques in the following questions.

7.1 Questions

Question 29: This question is on gravity tunnels.

(a) Find the time for a person to travel through a tunnel through the centre of the Earth, which has a radius of R (Figure 6.5).

(b) A man wants to build a straight tunnel (through the upper mantle) from London to New York City. Find the theoretical time to travel from a city to another using only gravity (Figure 7.1).

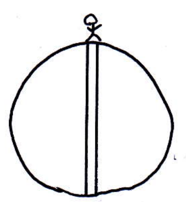


Figure 7.1: Tunnel through centre of the Earth.

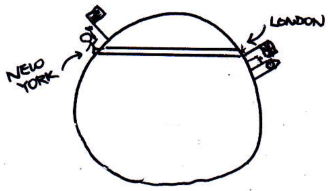


Figure 7.2: Tunnel from London to NYC.

Question 30: A spaceship is orbiting around a planet of radius R in a circular orbit of radius $3R$. The captain of the spaceship wants to orbit the planet in a new circular orbit of radius $9R$.

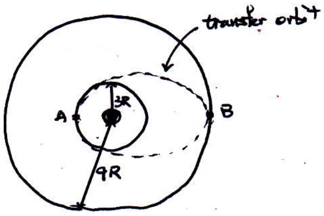


Figure 7.3: Spaceship Transfer.

(a) The most energy-efficient transfer is for the spaceship to transfer to an elliptical orbit known as the Hohmann Transfer.

   (i) Name points A and B. What is the relationship of vectors **v** and **r** there?

   (ii) Calculate the semi-major axis of the orbit.

   (iii) Calculate the initial velocity of the spaceship at A.

(b) In order to transfer to this elliptical orbit a tangential boost is required. Calculate this velocity boost (velocity change).

(c) Calculate the velocity of the spaceship at B, Hence calculate the velocity boost required at B.

(d) Find the transfer time. You may assume that the boost requires a negligible amount of time.

Question 31: This question is about perturbations on a circular orbit.

(I) Tangential Perturbations

(a) Consider a spaceship of mass m orbiting around a planet of mass M and radius R. Find the velocity of the spaceship if the orbit is circular and of radius $4R$.

(b) A tangential impulse is given to the spaceship such that it collapses to an orbit which just grazes the surface of the planet. Find this impulse.

(c) Now consider the spaceship given a general tangential impulse at point Q on the circular orbit. Draw the resulting orbits if

(i) The impulse is directed in the same direction as the orbiting velocity.

(ii) The impulse is directed in the opposite direction as the orbiting velocity.

Mark point Q and the two apses of the orbit (the pericentre and apocentre).

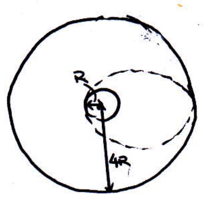


Figure 7.4: Radial impulse causes the spaceship to collapse to an orbit that just grazes the surface of the planet.

(II) Radial Perturbations

(a) The orbit is given a small radial impulse. Find the oscillation period of the resulting oscillation caused by the perturbation.

(b) Now suppose the impulse is finitely large. Draw the resulting orbit, marking the locations of the pericentre, apocentre and the point where the impulse is given.

(III) Out-of-plane Perturbations

(a) The spaceship is given an impulse perpendicular to the plane of motion. Argue why the original motion is contained in a subspace of a 3D Euclidean space. Describe the new orbit.

(b) Let's suppose that the impulse is continuously given (by, say, the engine of the spaceship). Calculate the torque given to the spaceship for the orbit to return to the same position as the spaceship completes a cycle of revolution of the orbit.

Question 32: A spaceship approaches a planet with initial velocity v_0 (measured far away from the planet of radius R). It just grazes the planet and flies away. Calculate the initial perpendicular distance between the centre of the planet and the spaceship velocity (indicated as the impact parameter b in Figure 7.5).

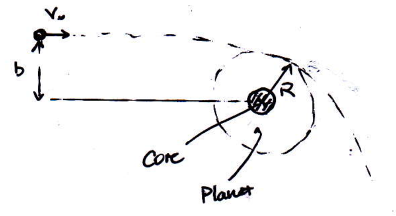


Figure 7.5: A spaceship with initial velocity v_0 and impact parameter b travelling in a hyperbolic orbit.

7.2 Solutions and Commentary

Question 29: Find the time for a person to travel through a gravity tunnel through the centre of the Earth, which has a radius of R (Figure 6.5). A man wants to build a straight tunnel (through the upper mantle) from London to New York City. Find the theoretical time to travel from a city to another using only gravity (Figure 7.1).
Solution: First we find oscillation period for a gravity tunnel through the centre of the Earth. Consider the man when he was distance r from the centre of the Earth. Using the Shell theorem, we know that spherical shells larger than radius r do not contribute to the restoring force, so the effective mass contributing to the restoring force is

$$M_{\text{eff}} = \frac{r^3}{R^3} M \tag{7.12}$$

where M is the mass of the Earth. Then Newton's Second Law gives

$$-\frac{GMm}{r^2}\frac{r^3}{R^3} = m\ddot{r} \tag{7.13}$$

Hence we have

$$\ddot{r} = -\frac{GM}{R^3} r \tag{7.14}$$

Comparing this with the SHM equation (see §8) gives the angular frequency as

$$\omega_0^2 = \frac{GM}{R^3} \tag{7.15}$$

The period is therefore

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \tag{7.16}$$

Therefore the time of travel is just half the period.

$$t = \pi\sqrt{\frac{R^3}{GM}} \tag{7.17}$$

Now consider a gravity tunnel not passing through the centre of the Earth (Figure 7.6)

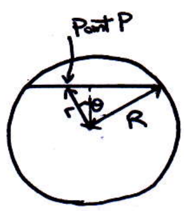


Figure 7.6: Gravity tunnel connecting NYC and London, with quantities included in the diagram.

Consider the person at point P of the track (distance r) from the centre. We apply the argument above (using the Shell Theorem) and Newton's Second Law gives

$$m\ddot{x} = -\frac{GM}{r^2}\frac{r^3}{R^3}m\sin\theta = -\frac{GMm}{R^3}r\sin\theta \tag{7.18}$$

where $\ddot{x}$ is the acceleration along the tunnel, with x measured from the middle of the tunnel (the equilibrium point). Since $x = r\sin\theta$, we write

$$\ddot{x} = -\frac{GM}{R^3}x \tag{7.19}$$

The angular frequency is

$$\omega_0^2 = \frac{GM}{R^3} \tag{7.15}$$

same as before. Therefore the time to travel from one city to the next is the same as before!

$$t = \pi\sqrt{\frac{R^3}{GM}} \tag{7.17}$$

Question 30: A spaceship is orbiting around a planet of radius R in a circular orbit of radius $3R$. The captain of the spaceship wants to orbit the planet in a new circular orbit of radius $9R$ (Figure 7.3). The most energy-efficient transfer is for the spaceship to transfer to an elliptical orbit known as the Hohmann Transfer. In order to transfer to this elliptical orbit a tangential boost is required. Calculate this velocity boost (velocity change). Then calculate the velocity of the spaceship at B, Hence calculate the velocity boost required at B. Finally find the transfer time. You may assume that the boost requires a negligible amount of time.
Solution: First we need find the initial velocity at A, i,e. the orbiting velocity of the lower orbit. This is a circular orbit so we can write

$$\frac{GM}{(3R)^2} = \frac{v_0^2}{3R} \tag{7.20}$$

This gives

$$v_0^2 = \frac{GM}{3R} \tag{7.21}$$

To find the boost required, we need to calculate the velocity at A when the spaceship is travelling in the elliptical orbit. The semi-major axis of the transfer orbit is

$$a = \frac{1}{2}(3R + 9R) = 6R \tag{7.22}$$

The energy in the orbit is

$$E = -\frac{GMm}{12R} = \frac{1}{2}mv_A^2 - \frac{GMm}{3R} \tag{7.23}$$

Solving for v_A^2 gives

$$v_A^2 = \frac{GM}{2R} \tag{7.24}$$

Hence the boost required is

$$\Delta v = v_A - v_0 = \sqrt{\frac{GM}{2R}} - \sqrt{\frac{GM}{3R}} \tag{7.25}$$

Now we calculate the velocity at B. Using the total energy again, we have

$$-\frac{GMm}{12R} = \frac{1}{2}mv_B^2 - \frac{GMm}{9r} \tag{7.26}$$

Solving for v_B^2 gives

$$v_B^2 = \frac{GM}{18R} \tag{7.27}$$

Now the velocity required to travel in the upper circular orbit is

$$v_u^2 = \frac{GM}{9R} \tag{7.28}$$

Therefore the boost required is

$$\Delta v = v_u - v_B = \frac{1}{3}\sqrt{\frac{GM}{R}}\left(1 - \frac{1}{\sqrt{2}}\right) \tag{7.29}$$

The period of the transfer orbit can be found using Kepler's Third Law. Mathematically this is

$$T^2 = \frac{4\pi^2}{GM}a^3 \tag{7.30}$$

The transfer time is half the period, giving

$$t = \sqrt{\frac{216\pi^2 R^3}{GM}} \tag{7.31}$$

Question 31: This question is about perturbations on a circular orbit.

(I) Tangential Perturbations

(a) Consider a spaceship of mass m orbiting around a planet of mass M and radius R. Find the velocity of the spaceship if the orbit is circular and of radius $4R$. A tangential impulse is given to the spaceship such that it collapses to an orbit which just grazes the surface of the planet. Find this impulse.

(b) Now consider the spaceship given a general tangential impulse at point Q on the circular orbit. Draw the resulting orbits if

(i) The impulse is directed in the same direction as the orbiting velocity.

(ii) The impulse is directed in the opposite direction as the orbiting velocity.

Mark point Q and the two apses of the orbit (the pericentre and apocentre).

(II) Radial Perturbations

(a) The orbit is given a small radial impulse. Find the oscillation period of the resulting oscillation caused by the perturbation.

(b) Now suppose the impulse is finitely large. Draw the resulting orbit, marking the locations of the pericentre, apocentre and the point where the impulse is given.

(III) Out-of-plane Perturbations

(a) The spaceship is given an impulse perpendicular to the plane of motion. Argue why the original motion is contained in a subspace of a 3D Euclidean space. Describe the new orbit.

(b) Let's suppose that the impulse is continuously given (by, say, the engine of the spaceship). Calculate the torque given to the spaceship for the orbit to return to the same position as the spaceship completes a cycle of revolution of the orbit.

Solution: This is an interesting problem on perturbations on circular orbits[2]. This involves drawing orbits - a skill that previously you might not have practised before.

Tangential Perturbations First we need to find the orbiting velocity of the original circular orbit. The centripetal acceleration is

$$\frac{GM}{(4R)^2} = \frac{v^2}{4R} \tag{7.32}$$

Solving for v^2 gives

$$v^2 = \frac{GM}{4R} \tag{7.33}$$

The new elliptical orbit has an semi-major axis of $a = \frac{1}{2}(R+4R)$. The total energy in the orbit is:

$$E = -\frac{GMm}{5R} = \frac{1}{2}mv^2 - \frac{GMm}{4R} \tag{7.34}$$

where

$$v^2 = \frac{GM}{10R} \tag{7.35}$$

Then the impulse given is

$$\Delta J = m\Delta v = m\sqrt{\frac{GM}{R}}\left(\frac{1}{2} - \frac{1}{\sqrt{10}}\right) \tag{7.36}$$

To draw the orbits, note that tangential perturbations given along the same direction of the orbit increases the velocity (and hence kinetic energy) at the point. This point has the largest velocity of the entire motion - the pericentre of the orbit. The apocentre of the orbit must be on the other side of the planet (Figure 7.7). The reverse is similar - when the tangential perturbation is given in the opposite direction of the orbiting velocity, then this point has the smallest velocity of the motion and is the apocentre of the orbit. The pericentre of the orbit is on the other side of the planet (Figure 7.8).

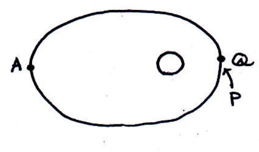


Figure 7.7: The resulting orbit when a tangential perturbation is given at point Q along the orbiting direction. The points labelled A and P are the apocentre and pericentre of the orbit.

[2]This is in fact one of my favourite problems in the sheet.

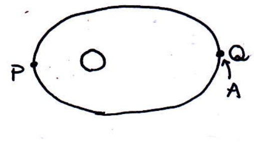


Figure 7.8: The resulting orbit when a tangential perturbation is given at point Q against the orbiting direction. The points labelled A and P are the apocentre and pericentre of the orbit.

Radial Perturbations The first part of the question involves finding the oscillation period resulted from a radial perturbation. This is found easily by considering the effective potential energy as derived in Equation 7.10 (and setting $A = GMm$).

$$U_{\text{eff}} = \frac{J^2}{2mr^2} - \frac{A}{r} \tag{7.10}$$

We need to first find the equilibrium of the radial motion. This is easily found by setting $\frac{dU}{dr} = 0$.

$$U'_{\text{eff}} = -\frac{J^2}{mr^3} + \frac{A}{r^2} = 0 \tag{7.37}$$

This solves to give

$$r_0 = \frac{J^2}{mA} \tag{7.38}$$

This value of r_0 is called the semi-latus rectum of the motion. This is the length of a line from the focus to the orbit where the line is parallel to the minor-axis. The semi-latus rectum only depends on the angular momentum and the masses involved in the motion. Further differentiating gives

$$U''_{\text{eff}} = \frac{3J^2}{mr^4} - \frac{2A}{r^3} \tag{7.39}$$

Substituting the equilibrium value of $r = r_0$ gives:

$$U''_{\text{eff}} = \frac{GMm}{r_0^3} \tag{7.40}$$

In a general potential well, the second derivative of the potential energy expression about the equilibrium gives the effective spring constant to the motion[3]. This therefore gives the oscillation frequency as:

$$\omega_0^2 = \frac{GM}{r_0^3} \tag{7.41}$$

where r_0 is the semi-latus rectum. Note that a radial impulse does not change the angular momentum, therefore leaving the semi-latus rectum unchanged. Since

[3]This is most easily checked by Taylor-expanding near the equilibrium of the potential well, and noting the $U'_{\text{eff}} = 0$. The argument is explored in the appendix.

the semi-latus rectum of a circular orbit is the radius of the orbit, the impact point must be the semi-latum point on the orbit. Sketching in the pericentre and apocentre of the orbit gives the resulting orbit.

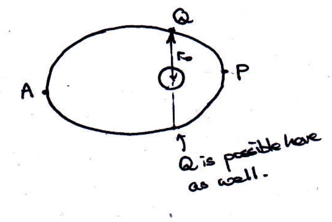


Figure 7.9: The resulting orbit when a radial perturbation is given at point Q (here the perturbation is directed radially outwards). The points labelled A and P are the apocentre and pericentre of the orbit.

Perpendicular Perturbations In providing a perpendicular impulse, the angular momentum vector of the orbit is changed. The circular motion is still contained in a plane (2D subspace of 3D), but the plane of motion is tilted from the original motion.

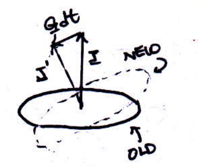


Figure 7.10: The resultant orbit when a perpendicular impulse was given.

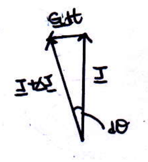


Figure 7.11: Change in the angular vector when providing an impulse of size Gdt. The resulting angular momentum vector has the same magnitude as before but is pointing in a different direction.

If the impulse is continuously given, we can denote the torque given to the system as τ and use the vector diagram drawn in Figure 7.9 to give

$$J d\theta = \tau dt \tag{7.42}$$

which gives

$$\omega_p = \frac{\tau}{J} \tag{7.43}$$

This is precisely the precessing frequency of the system (the system is is undergoing slow precession). Equating Equations 7.41 and 7.43 gives

$$\tau^2 = \frac{GM}{R^3} J^2 \tag{7.44}$$

where

$$J^2 = (mrv)^2 = GMm^2R \tag{7.45}$$

Therefore this gives

$$\tau = \frac{GMm}{R} \tag{7.46}$$

Question 32: A spaceship approaches a planet with initial velocity v_0 (measured far away from the planet of radius R). It just grazes the planet and flies away. Calculate the initial perpendicular distance between the centre of the planet and the spaceship velocity (indicated as the impact parameter b in Figure 7.5).
Solution: This is a hyperbolic orbit, a type of open orbits that you might not have seen before. However, conservation laws still holds in these type of orbits. First begin with the conservation of angular momentum.

$$mv_0b = mvR \tag{7.47}$$

where b is the impact parameter, the perpendicular distance between the initial vector (assuming infinitely far away) and the planet. The velocity and radial vectors are perpendicular at the closest point of approach. Solving for v gives

$$v = \frac{v_0 b}{R} \tag{7.48}$$

Conservation energy in the system gives (assuming the initial position of the object is infinitely far away, so the potential energy is zero)

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{GMm}{R} \tag{7.49}$$

Substituting Equation 7.48 gives

$$v_0^2 = v^2 - \frac{2GM}{R} \tag{7.50}$$

Solving for b gives

$$b = \frac{R}{v_0}\sqrt{v_0^2 + \frac{2GM}{R}} \tag{7.51}$$

7.3 Take-home skills

1. Using conservation laws in gravitational problems.
2. Using Shell Theorem in problems.
3. Drawing orbits by using physical arguments.
4. Dealing with oscillations in a general potential well.
5. Dealing with precession problems.

Chapter 8

Simple Harmonic Motion

A harmonic motion is a type of *periodic motion* that repeats itself in a given set of period. This type of motion can be found anywhere from mechanical to electrical situations. It is also the fundamental concept in *wave optics*. Examples of harmonic motion include pendulums and alternate circuits. A simple harmonic motion is a type of harmonic motion:

Definition 8.1. *A **simple harmonic motion** is a special kind of harmonic motion for which when the system is slightly disturbed from its equilibrium position, a restoring force will provide an acceleration proportional to the displacement to return it to its equilibrium position. Mathematically this is expressed as*

$$\ddot{x} = -\omega^2 x \tag{8.1}$$

where ω is the angular frequency of the motion.

The solution of this ordinary differential equation (see [1]) is

$$x(t) = C\cos(\omega t) + D\sin(\omega t) \tag{8.2}$$

where C and D are constants fixed by the initial conditions. This can be written also in the form

$$x(t) = A\cos(\omega t + \phi) \tag{8.3}$$

where A is the amplitude and ϕ is the phase factor fixed by the initial conditions. Finding the equation of motion, identifying the frequency of oscillation and using the test solution is sufficient in answering the following questions. Other results related to simple harmonic motion that are not central to the argument can be found in [2].

8.1 Questions

Question 33: Calculate the frequency of oscillation for the following systems.

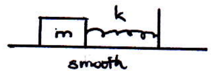


Figure 8.1: A spring-mass system. The spring constant is k and the mass is m.

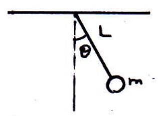


Figure 8.2: A simple pendulum made of a string of length L and a bob of mass m.

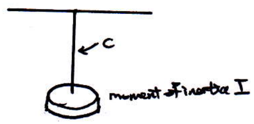


Figure 8.3: A torsional pendulum. The moment of inertial of the pendulum is I and the torsional coefficient of the wire is c.

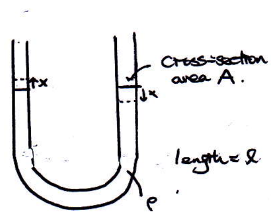


Figure 8.4: A U-tube filled with liquid of density ρ. The cross-sectional area of the tube is A and the total length of the liquid in the tube is l.

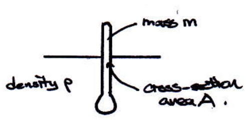


Figure 8.5: A hydrometer floating on the surface of a liquid of density ρ. It is oscillating vertically,. The cross-sectional area of the tube part is A and the mass of the hydrometer is m.

Question 34: A uniform plank of length L is put on two inwardly rotating poles initially distance $\frac{l}{2}$ from the centre of mass of the plank. Suppose the plank is displaced by a distance x.

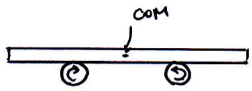


Figure 8.6: The plank on two rotating poles system.

(a) Calculate the normal forces on the two poles.

(b) Hence find the restoring force.

(c) Find the period of oscillation.

Question 35: A good model to atomic oscillations is given by the Lennard-Jones potential (sketched in Figure 8.7).

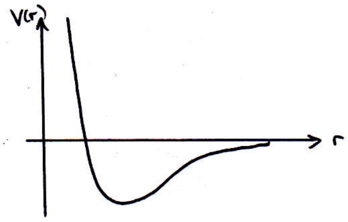


Figure 8.7: Sketch of the Lennard-Jones potential.

$$U = -\frac{A}{r} + \frac{B}{r^9} \tag{8.4}$$

where $A = \frac{Ne^2}{4\pi\epsilon_0}$ and B is a constant. Here e is the electron charge, ϵ_0 is the permittivity of free space and N is the number of electrons. Consider the H-Cl bond. The equilibrium atomic separation is $r = 0.13\text{nm}$. Find B and hence the frequency of vibration of the molecule.

Question 36: A question on springs.

(a) Prove that a system of springs in parallel has an effective spring constant of

$$k_{\text{eff}} = \sum_i k_i \tag{8.5}$$

(b) Prove that a system of springs in series has an effective spring constant of

$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i} \tag{8.6}$$

(c) Consider a box of mass $2m$ and initial velocity v_0 shot into a box of mass m (initially at rest) with an ideal spring of spring constant k sticking out on its side (Figure 8.8). After the mass touches the spring they are connected together for the rest of the motion. Describe quantitatively the resultant motion of the system.

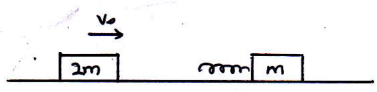


Figure 8.8: A box of mass $2m$ and initial velocity v_0 fired head-on to a box of mass m initially at rest. A spring of spring constant k is sticking out on the left side of the stationary box.

8.2 Solutions and Commentary

Question 33: Calculate the frequency of oscillation for the following systems.

1. A spring-mass system. (Figure 8.1)
2. A simple pendulum. (Figure 8.2)
3. A torsional pendulum. (Figure 8.3)
4. A U-tube. (Figure 8.4)
5. A hydrometer. (Figure 8.5)

Solution: In these equations all we have to do is to find the equation of motion of the system.

Spring-mass System This is trivial. Applying NII gives

$$m\ddot{x} = -kx \tag{8.7}$$

Comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{k}{m} \tag{8.8}$$

Simple Pendulum Here use the rotational form of NII. This gives

$$I\ddot{\theta} = -mgl\sin\theta \tag{8.9}$$

Using the small angle approximation $\theta \approx \sin\theta$ and noting that the moment of inertia of the system is $I = mr^2$, this gives

$$\ddot{\theta} = -\frac{g}{l}\theta \tag{8.10}$$

Comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{g}{l} \tag{8.11}$$

Torsional Pendulum This is the rotational form of the spring-mass system. The equation of motion is

$$I\ddot{\theta} = -c\theta \tag{8.12}$$

Comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{c}{I} \tag{8.13}$$

where c is the torsional spring coefficient and I is the moment of inertia of the pendulum.

U-Tube Here when the two sides of the tube have unequal heights of liquid of density ρ, the net restoring force is twice the weight of the displaced liquid of height x. This weight is

$$W = 2\rho g A x \tag{8.14}$$

where A is the cross-sectional area of the tube. NII of the liquid (the entire length) now gives

$$\rho l \ddot{x} = -2\rho g x \tag{8.15}$$

Comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{2g}{I} \tag{8.16}$$

Hydrometer The restoring force is provided by the buoyant force of the displaced hydrometer length. NII gives

$$m\ddot{x} = -\rho A x g \tag{8.17}$$

where m is the mass of the hydrometer, ρ is the density of the liquid, A is the cross-sectional area of the hydrometer and g is the gravitational constant. Comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{\rho A g}{m} \tag{8.18}$$

Question 34: A uniform plank of length L is put on two inwardly rotating poles initially distance $\frac{l}{2}$ from the centre of mass of the plank. Suppose the plank is displaced by a distance x (Figure 8.6). Find the period of oscillation of the motion.

Solution: Begin with drawing a free-body diagram of the problem.

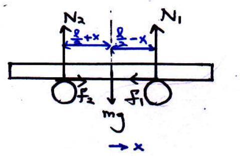


Figure 8.9: Free-body diagram of the plank system.

Taking moments about points A and B gives

$$N_1 = \frac{mg}{l}\left(\frac{l}{2} + x\right) \tag{8.19}$$

$$N_2 = \frac{mg}{l}\left(\frac{l}{2} - x\right) \tag{8.20}$$

Restoring force is given by

$$F_{\text{res}} = f_2 - f_1 = \mu_k(N_2 - N_1) \tag{8.21}$$

Using Equations 8.19 and 8.20 gives

$$m\ddot{x} = -\frac{2mg\mu_k}{l}x \tag{8.22}$$

Therefore comparing with the SHM equation (Equation 8.1) gives

$$\omega_0^2 = \frac{2g\mu_k}{l} \tag{8.23}$$

giving

$$T = 2\pi\sqrt{\frac{l}{2g\mu_k}} \tag{8.24}$$

Question 35: A good model to atomic oscillations is given by the Lennard-Jones potential (sketched in Figure 8.7).

$$U = -\frac{A}{r} + \frac{B}{r^9} \tag{8.25}$$

where $A = \frac{Ne^2}{4\pi\epsilon_0}$ and B is a constant. Here e is the electron charge, ϵ_0 is the permittivity of free space and N is the number of electrons. Consider the H-Cl

bond. The equilibrium atomic separation is $r = 0.13$nm. Find B and hence the frequency of vibration of the molecule.
Solution: This is similar to finding the frequency of oscillation of the radially perturbed orbit. The equilibrium is found by setting $U' = 0$.

$$\frac{dU}{dr} = \frac{A}{r^2} - \frac{9B}{r^{10}} \tag{8.26}$$

This solves to give

$$B = \frac{Ar_0^8}{9} \tag{8.27}$$

where r_0 is the equilibrium bond length. The second derivative of the potential energy gives the effective spring constant.

$$U''(r_0) = -\frac{2A}{r_0^3} + 10\frac{9B}{r_0^{11}} = \frac{8A}{r_0^3} \tag{8.28}$$

The frequency oscillation is therefore

$$\omega_0^2 = \frac{8A}{\mu r_0^3} \tag{8.29}$$

where μ is the reduced mass defined as

$$\frac{1}{\mu} = \frac{1}{m_{\mathrm{H}}} + \frac{1}{m_{\mathrm{Cl}}} \tag{8.30}$$

Substituting the values in given in the question gives

$$\nu \approx 1.1 \times 10^{14}\mathrm{Hz} \tag{8.31}$$

Question 36: A question on springs.

(a) Prove that a system of springs in parallel has an effective spring constant of

$$k_{\text{eff}} = \sum_i k_i \tag{8.32}$$

(b) Prove that a system of springs in series has an effective spring constant of

$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i} \tag{8.33}$$

(c) Consider a box of mass $2m$ and initial velocity v_0 shot into a box of mass m (initially at rest) with an ideal spring of spring constant k sticking out on its side (Figure 8.8). After the mass touches the spring they are connected together for the rest of the motion. Describe quantitatively the resultant motion of the system.

Solution: For a system of springs connected in parallel, the extension length is constant in all springs. This gives

$$F = \sum_i k_i x = x \sum_i k_i = k_{\text{eff}} x \tag{8.34}$$

Rearranging gives

$$k_{\text{eff}} = \sum_i k_i \tag{8.32}$$

For a system of springs connected in series, the force is constant throughout. Note that

$$x = \sum_i x_i \tag{8.35}$$

Hence

$$\frac{F}{k_{\text{eff}}} = \sum_i \frac{F}{k_i} \tag{8.36}$$

Eliminating F gives

$$\frac{1}{k_{\text{eff}}} = \sum_i \frac{1}{k_i} \tag{8.33}$$

In the following spring-mass system, the resulting motion is a combination of linear motion and oscillation. The entire system travels at the net velocity at the centre-of-mass velocity:

$$v_{\text{COM}} = \frac{2mv_0}{2m+m} = \frac{2}{3}v_0 \tag{8.37}$$

What remains to find is the oscillation frequency. The masses performs simple harmonic motion about the centre-of-mass. Suppose the spring constant connecting to mass m is k_1 and the one connecting the mass $2m$ is k_2. For springs connecting in series this is

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \tag{8.38}$$

Note that the frequency of oscillation of the two masses must be the same, otherwise there will be infinite acceleration at the centre-of-mass. This leaves

$$\omega_0^2 = \frac{k_1}{m} = \frac{k_2}{2m} \tag{8.39}$$

This leaves

$$k_2 = 2k_1 \tag{8.40}$$

Combining Equations 8.38 and 8.40 gives

$$k_1 = \frac{3}{2}k \tag{8.41}$$

Hence the oscillation frequency is

$$\omega_0^2 = \frac{3k}{2m} \tag{8.42}$$

The general motion of the masses is

$$x_i = A_i \cos(\omega t + \phi_i) + v_{\mathrm{COM}}t \tag{8.43}$$

where $i = 1, 2$ are labels for the two particles and A_i and ϕ_i are determined by the initial conditions (which I am not bothered to solve). This is a normal mode problem with the two orthogonal normal modes being the simple transverse mode and oscillating mode.

8.3 Take-home Skills

1. Finding equation of motions of the system via either the energy method or Newton's Second Law.
2. Using results with regards to a general potential well.
3. Dealing with combinations of springs.

Appendix A

Basics of Lagrangian Mechanics

Lagrangian Mechanics is a reformulation of Classical Mechanics. There is no new physics - it is simply a reformulation of a basic mechanics. The key to the method is to solve the Euler-Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{A.1}$$

This is typically derived by introducing the action integral, for which we will define below.

Definition A.1. *The action of a motion $\mathcal{S}$ is defined as the time integral of a function.*

$$\mathcal{S} = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt \tag{A.2}$$

where the general coordinates of the system is described as q_i, with the subscript i indicating the numbering of a multi-particle system. Here the function $L(q, \dot{q}, t)$ is called the Lagrangian and is defined as

$$L = T - V \tag{A.3}$$

where T and V are the kinetic and potential energies respectively.

We will not go into the details of the origin of the Lagrangian function (see for details). However, we will derive the Euler-Lagrange equation from Hamilton's Principle stated as follows:

Proposition A.1. ***Hamilton's Principle*** *states that for a motion in classical mechanics action $\mathcal{S}$ is stationary for small variations $\delta q_i(t)$ about path for all q_i.*

This is also known as the *Principle of Least Action*. The problem now becomes a variational problem[1] with the end points of the integral fixed. Taking a small variation $q_i \to q_i + \delta q_i$ gives:

$$\delta \mathcal{S} = \int_{t_1}^{t_2} \left(\delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \tag{A.4}$$

where Einstein's summation convention is used[2]. Performing an integration-by-parts give

$$\delta \mathcal{S} = \left[\delta q_i \frac{\partial L}{\partial \dot{q}_i} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] = 0 \tag{A.5}$$

The first term vanishes since δq_i is unchanged with fixed end points. The second term must remain vanished under an arbitrary change in δq_i, therefore the integrand must be zero. This gives the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \tag{A.1}$$

An effective use of the Lagrangian method must involve picking the correct generalised coordinates. This is normally done by considering the constraints of the system and eliminating any additional degrees of freedom. For a full account for the method, please refer to [3] .

[1] This has to do with maximising or minimising functionals, in this case $\mathcal{S}$. For details see [3] and [4].

[2] This means repeated indices are implicitly summed.

Appendix B

Three Balls Problem

This section concerns with the solution plotted out in The Three Balls Problem in §5. Here I will provide my solution to the problem.

Lagrangian The key to success is to write down the correct Lagrangian. Here the two coordinates I have used is x, the horizontal displacement of the entire system and θ, the angle between the horizontal axis and the string, as defined in Figure 5.8. The kinetic and potential energies are

$$T = \frac{1}{2}m\dot{x}^2 + m\left[\left(\dot{x}\sin\theta + l\dot{\theta}\right)^2 + \left(\dot{x}\cos\theta\right)^2\right] \tag{B.1}$$

$$V = 0 \tag{B.2}$$

Therefore the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + m\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\sin\theta + \dot{x}^2\right) \tag{B.3}$$

This is defined in Mathematica as the following

```
Lagrangian =
  1/2 m*D[x[t], t]^2 +
   m*((l*D[\[Theta][t], t] +
        D[x[t], t]*Sin[\[Theta][t]])^2 + (D[x[t], t]*
        Cos[\[Theta][t]])^2)
```

Equation of Motion Now to obtain the equation of motion we use the Euler-Lagrange Equation. The Mathematica code required is:

```
EQNLana = D[D[Lagrangian, D[x[t], t]], t] == D[Lagrangian, x[t]];
EQNLanb =
  D[D[Lagrangian, D[\[Theta][t], t]], t] ==
   D[Lagrangian, \[Theta][t]];
```

The Lagrangian is independent in x so the conserved quantity is found by

```
EQNCons = Simplify[D[Lagrangian, D[x[t], t]]] == B;
```

So $\dot{x}$ can be found by the command

```
VelSol = Flatten[Solve[EQNCons, D[x[t], t]]];
```

The Equation of Motion is therefore

```
EQNLanbtosolve = Simplify[EQNLanb /. VelSol /. D[VelSol, t]];
```

which gives

$$\ddot{\theta} = \frac{2\cos\theta\sin\theta}{1+2\cos^2\theta}\dot{\theta}^2 \tag{B.4}$$

exactly as obtained via considering forces.

Numerical Differentiation The equation of motion has no closed-form solutions. Mathematica has a built in function to solve numerical integrals. I have then manually implemented the boundary condition (collision) by finding the root of the problem at the respective angles and implementing $\dot{\theta}|_+ = -\dot{\theta}|_-$ at the boundaries. The following is the code to generate the two graphs.

```
BCLan1 = \[Theta][0] == \[Pi]/6;
BCLan2 = \[Theta]'[0] == -0.1;
ConstantsRule = {A -> 1, B -> 0, l -> 1, m -> 1};
SolLan2 =
  Flatten[NDSolve[
    EQNLan2btosolve && BCLan1 && BCLan2 /.
     ConstantsRule, \[Theta], {t, 0, 20}]];
Plot[\[Theta][t] /. SolLan2, {t, 0, 20}, PlotStyle -> Red,
  AxesOrigin -> {0, 0}, PlotRange -> All];
Root1 = FindRoot[\[Theta][t] /. SolLan2, {t, 2, 5}];
Plot[\[Theta]'[t] /. SolLan2, {t, 0, 20}, PlotStyle -> Red,
  AxesOrigin -> {0, 0}, PlotRange -> All];
BCLan3 = (\[Theta]'[t] /. Root1) == (-\[Theta]'[t] /. Root1 /.
     SolLan2);
BCLan4 = (\[Theta][t] /. Root1) == (\[Theta][t] /. Root1 /. SolLan2);
SolLan2a =
  Flatten[NDSolve[
    EQNLan2btosolve && BCLan3 && BCLan4 /.
     ConstantsRule, \[Theta], {t, t /. Root1, 50}]];
```

```
Show[Plot[\[Theta][t] /. SolLan2a, {t, t /. Root1, 50},
   PlotStyle -> Red, AxesOrigin -> {0, 0}, PlotRange -> All],
  Plot[\[Theta][t] /. SolLan2, {t, 0, t /. Root1}, PlotStyle -> Blue,
   AxesOrigin -> {0, 0}, PlotRange -> All]];
Show[Plot[\[Theta]'[t] /. SolLan2a, {t, t /. Root1, 50},
   PlotStyle -> Red, AxesOrigin -> {0, 0}, PlotRange -> All],
  Plot[\[Theta]'[t] /. SolLan2, {t, 0, t /. Root1}, PlotStyle -> Blue,
    AxesOrigin -> {0, 0}, PlotRange -> All]];
Root2 = FindRoot[(\[Theta][t] - \[Pi]) /. SolLan2a, {t, 30, 40}];
BCLan5 = (\[Theta]'[t] /. Root2) == (-\[Theta]'[t] /. Root2 /.
     SolLan2a);
BCLan6 = (\[Theta][t] /. Root2) == (\[Theta][t] /. Root2 /.
     SolLan2a);
SolLan2b =
  Flatten[NDSolve[
    EQNLan2btosolve && BCLan5 && BCLan6 /.
     ConstantsRule, \[Theta], {t, t /. Root2, 100}]];
Root3 = FindRoot[\[Theta][t] /. SolLan2b, {t, 58, 63}]
Show[Plot[\[Theta][t] /. SolLan2b, {t, t /. Root2, t /. Root3},
  PlotStyle -> Green, AxesOrigin -> {0, 0}, PlotRange -> All],
 Plot[\[Theta][t] /. SolLan2a, {t, t /. Root1, t /. Root2},
  PlotStyle -> Red, AxesOrigin -> {0, 0}, PlotRange -> All],
 Plot[\[Theta][t] /. SolLan2, {t, 0, t /. Root1}, PlotStyle -> Blue,
  AxesOrigin -> {0, 0}, PlotRange -> All],
 AxesLabel -> {t, \[Theta]}]
Show[Plot[\[Theta]'[t] /. SolLan2b, {t, t /. Root2, t /. Root3},
  PlotStyle -> Green, AxesOrigin -> {0, 0}, PlotRange -> All],
 Plot[\[Theta]'[t] /. SolLan2a, {t, t /. Root1, t /. Root2},
  PlotStyle -> Red, AxesOrigin -> {0, 0}, PlotRange -> All],
 Plot[\[Theta]'[t] /. SolLan2, {t, 0, t /. Root1}, PlotStyle -> Blue,
  AxesOrigin -> {0, 0}, PlotRange -> All], AxesLabel -> {t, \[Omega]}]
```

Appendix C

On central force orbits

In this section some previously assumed results will be derived and proved.

C.1 Conic forms of orbits

We begin by deriving the orbits equation, which allows us to derive the solution that gives the form of the orbits. Begin by considering the total energy of the problem

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{A}{r} \tag{C.1}$$

where J is the angular momentum of the system and $A = GMm$. Now we use the substitution $u = \frac{1}{r}$, and by noting that

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = -\dot{\phi}r^2\frac{du}{d\phi} \tag{C.2}$$

Then

$$\dot{r} = -\frac{J}{m}\frac{du}{d\phi} \tag{C.3}$$

Substituting into Equation C.1 gives

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2m}{J}(E + Au) = 0 \tag{C.4}$$

Completing the square gives

$$\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{r_0}\right)^2 - \left(\frac{1}{r_0^2} + \frac{2mE}{J^2}\right) = 0 \tag{C.5}$$

where r_0 is the semi-latus rectum (see later) given by

$$r_0 = \frac{J^2}{mA} \tag{C.6}$$

we define e as a constant as follows:

$$e^2 = 1 + \frac{2mEr_0^2}{J^2} = 1 + \frac{2EJ^2}{mA^2} \tag{C.7}$$

The differential equation becomes

$$\left(\frac{du}{d\phi}\right)^2 = \frac{e^2}{r_0^2} - \left(u - \frac{1}{r_0}\right) \tag{C.8}$$

This has a solution

$$u = \frac{1}{r_0}(1 + e\cos(\phi - \phi_0)) \tag{C.9}$$

Reorienting the axes allow we to set $\phi_0 = 0$. This gives the conic section equation

$$r = \frac{r_0}{1 + e\cos\phi} \tag{C.10}$$

This shows that the orbits of central forces are conic sections with the parameter e, known as the eccentricity, determining the shape of the orbit.

C.2 Total energy in an orbit

Now we try and derive the total energy in a closed elliptical orbit.

$$E = -\frac{A}{2a} \tag{C.11}$$

We start by considering the energy at the perihelion. This is

$$E = \frac{1}{2}mv_P^2 - \frac{A}{r_P} \tag{C.12}$$

Note that at the perihelion, using Equation C.10 (set $\phi = 0$) and the definition of angular momentum i.e.

$$r_P = \frac{r_0}{1 + e} \tag{C.13}$$

$$L = mv_P r_P \tag{C.14}$$

The major-axis is

$$2a = \frac{r_0}{1 + e} + \frac{r_0}{1 - e} \tag{C.15}$$

Substituting this into Equation C.12 gives

$$E = -\frac{A}{2a} \tag{C.11}$$

or

$$E = -\frac{GMm}{2a} \tag{C.16}$$

as required.

Bibliography

[1] K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical methods for physics and engineering.* Cambridge University Press, 2006.

[2] I. G. Main, *Vibrations and waves in physics.* Cambridge Univ. Press, 2002.

[3] L. D. Landau and L. E. M., *Mechanics.* Pergamon Press, 1969.

[4] H. Goldstein, J. Safko, and C. P. Poole, *Classical mechanics.* Pearson, 2014.