Summer Course Preliminary Test

Time allowed: 2 hours
Max credit: 20 pts

24^{th} June, 2019

Read This Carefully:

This paper consists of 5 pages and 4 questions. Questions 1 to 3 are questions related to Classical Mechanics, each carrying 10 marks. Question 4 is related to Electrodynamics and it is worth 20 marks. The maximum credit is 20 marks. **You are not required to complete every question, and you may choose up to two questions to finish.** You may also do parts from other questions, and bonus marks will be added to your overall credit. The approximate mark of each question part is indicated on the right margin. Note that you are not expected to complete any questions and this test is only used to see how much you know so just do your best!

Question:	1	2	3	4	Total
Points:	10	10	10	20	50
Score:					

Question 1

(Inspired by one of Trinity's interview questions.) A perfectly spherical ball of mass m and radius a lies in a large empty cylinder of mass M and radius R. The cylinder is given an initial impulse of $\mathbf{J}$ to the right such that it starts rolling without slipping with a velocity $\mathbf{v}$.

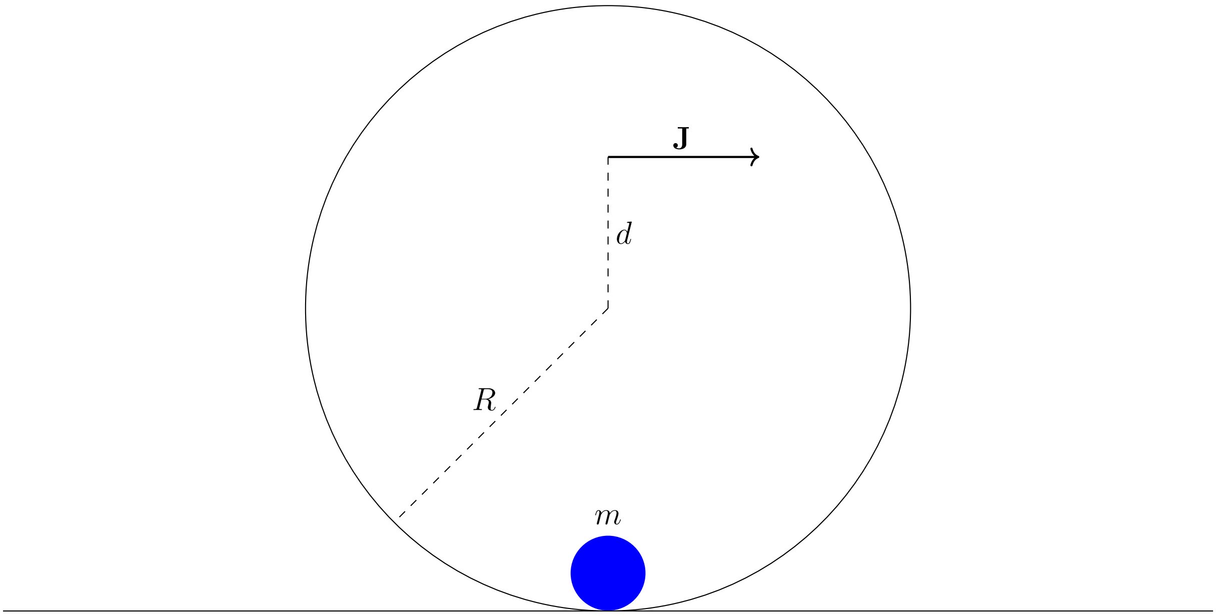

Figure 1: The cross-section of the considered cylinder perpendicular to the cylinder axis.

(a) Define the term *moment of inertia*. Describe how a general rigid body's motion can be analysed. (1)

(b) The moment of inertia of a thin disc is $I = \frac{1}{2}mr^2$ and that of a thin ring is $I = mr^2$. Write down the moment of inertia of the cylinder about its centre axis. Derive the moment of inertia of the spherical ball. (2)

(c) Suppose that the dimensions of the spherical ball can be neglected; i.e. $m << M$ and $a << R$. By considering the fact that the ball is rolling after the application of the small impulse, find the perpendicular distance d where the impulse is applied (See figure 1). Write down the initial velocity $\mathbf{v}$ of the cylinder in terms of $\mathbf{J}$ and M. (2)

(d) Draw a force diagram of the spherical ball **in the frame of the rotating cylinder** when the ball subtends an angle θ from the normal between the contacts of the cylinder and the underlying horizontal flat surface. You may have to also consider the direction of the friction. Describe qualitatively what happens to the ball if it is given an initial push. (2)

(e) Now assume that the cylinder is fixed to the ground and the internal surface of the cylinder is smooth. The spherical ball starts moving with the initial position as sketched in figure 1. The initial kinetic energy of the ball is E, and it is given that the ball does not reach the highest point of the cylinder. Find the constraint of E and the angular displacement travelled by the ball before it detaches from the cylindrical surface. You may assume that $a << R$. (3)

Question 2

(A question on differential equations and ODE applications in physics.)

(a) Define the term *ordinary differential equations.* Solve the following ODEs: (2)

i.
$$\frac{dy}{dx} + (\ln x + 1)y = x^{-x}$$

ii.
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4e^{-x}$$

(b) Consider a spring-mass system. Write down the equation of motion of the mass, and hence find the angular frequency of the simple harmonic motion. Using this as an example, hence describe what a simple harmonic motion is. (2)

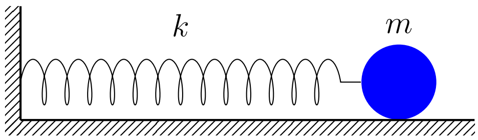

Figure 2: A ball of mass m is connected to the wall by a spring with a spring constant k. Assume that all surfaces are frictionless.

(c) Now suppose we have a system constrained in a general potential well. By Taylor expanding the potential function $V(x)$ at the point $x = x_{min}$, where x_{min} is where the minimum occurs, show that, for small perturbations, the dependence of potential is always a quadratic relationship. Hence find the effective mass of the system. (2)

(d) Suppose now the spring-mass system is put in a resistive fluid that gives a damping force $F_{damping} = -b\dot{x}$. By considering the energy change of the system, find the general equation of motion. Show that there are three damping regimes, and find the general solutions in each case. (3)

(e) Now suppose that $b = 2m$ and $k = m$. Given that the initial conditions of the ball are $x(0) = A$ and $\dot{x}(0) = 0$, find $x(t)$. (1)

Question 3

(Adapted from one of Cambridge's exam questions.)

(a) State the definition of a *Riemann Integral.* Evaluate the following integral, where a and b are constants. (2)

$$\int \frac{a}{(b^2 + z^2)^{\frac{3}{2}}} dz$$

(b) State *Newton's Law of Gravitation.* Sketch the lines of force (field lines) resulting from the presence of a large mass M. (1)

(c) Gauss' Theorem for gravitation states that for a body of mass M, the gravitational field can be calculated using: (1)

$$\oiint \mathbf{g} \cdot d\mathbf{S} = -CM$$

where C is a constant. By considering a spherical surface centred at the point object of mass M, find C.

(d) Using Gauss' Theorem, derive the gravitational field due to a infinitely long rod of density ρ. Make sure you have included a diagram in your derivation. (2)

(e) Consider the following finite rod of length $2a$. (2)

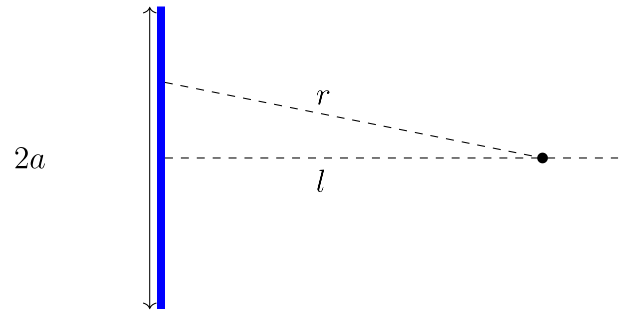

Figure 3: The considered finite rod of length $2a$ and mass m.

By considering infinitesimal masses separated from the middle line, compute the gravitational field a distance l from the centre of rod perpendicular to the axis of rod.

(f) Consider the cases where $l << a$ and $a << l$. Comment on the results. (2)

Question 4

(For the mathematically talented.)

(a) State the "definition" of the del operator in Cartesian coordinates. State, qualitatively, how the del operator looks in a general orthogonal curvilinear coordinate system. (1)

(b) Define *gradient, divergence and curl.* Also state *Divergence Theorem* and *Stoke's Theorem.* By considering the fact that Gauss' Law in electrostatics states that the surface integral of an electric field about an arbitrary surface is (2)

$$\oiint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{enc}}{\epsilon_0}$$

find the differential form of Gauss' Law using one of the integral theorems.

(c) Consider the differential $Pdx + Qdy = df$. By using chain rule of partial differentiation, find the condition for the differential to be *exact.* Hence state the condition for a general vector field $\mathbf{U}$ to be exact. (2)

(d) Suppose now you can express the electric field in terms of the gradient of a scalar field V as follows: (2)

$$\mathbf{E} = -\nabla V$$

Using the differential form of Gauss's Law, derive the equation relating the electric potential and the enclosed charge. Restate your answer where your arbitrary surface encloses no charges. This is known as Laplace's Equation.

(e) State the condition for two functions $f(x)$ and $g(x)$ to be orthogonal within the interval $[a, b]$. By considering the double angle formulae, show that (3)

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L} dx = \delta_{mn}$$

$$\frac{1}{L}\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = \delta_{mn}$$

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L} dx = 0 \quad \forall m, n$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

(f) Consider a general function that is expressed by: (2)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

By using the results in the previous part, derive the general formulae for the Fourier coefficients a_n and b_n.

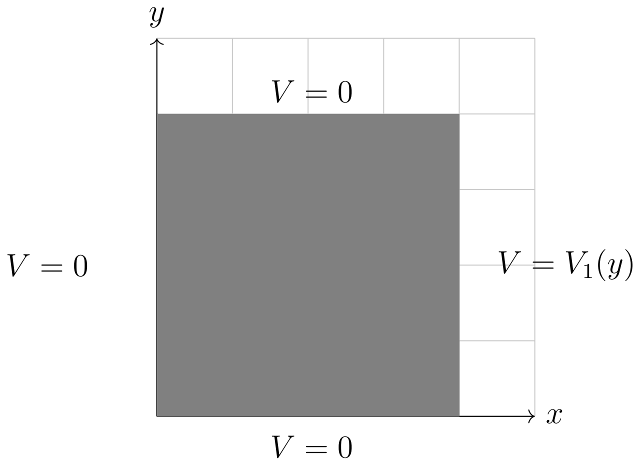

Figure 4: The considered square plate.

(g) Now consider a square plate of length a in a 2D Cartesian plane. There is no net charge in the conductor. Three of the ends of the square plate are earthed. One of the ends of the plate is connected to a potential generator with the function $V(y)$ at any time t.

i. Write down the partial differential equation describing the system. (1)

ii. By considering $V(x, y) = X(x)Y(y)$, where $X(x)$ and $Y(y)$ are functions of x and y only, write the differential equation into two equations. (2)

iii. Solve the differential equations. Using the boundary conditions, decide what your relating constant should be. Simplify your answers using the results your have derived above. Find the full solution of $V(x, y)$ when $V_1(y) = y$. (5)