Short Summer Project Report

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This document is a short summary for the summer project completed in the summer of 2020 in DAMTP. My supervisor is Dr Katarzyna Kowal and I worked on a theoretical project to analyse the stability of lubricated viscous gravity currents for non-Newtonian fluids. The project took approximately two months to finish (and it is ongoing at the moment when this report is written).

1 Overview

The project was planned in March 2020 to be an experimental project to work on an ongoing study with the stabilisation contributions from a lubricated till under glaciers. However, owing to the Covid-19 pandemic, the mentioned project could not be carried out as laboratories in Cambridge were closed and students were sent back home immediately following to conclusion of Lent 2020. Fortunately, I was given the opportunity to undergo a theoretical project with Trinity providing the necessary funding to investigate the theoretical elements of this ongoing study. The project aims to expand the findings in *Kowal and Worster (2019b)* to non-Newtonian and fluids and used *Mathematica* as the main coding tool for computational analysis. Similar procedures illustrated in *Kowal and Worster (2019b)* were taken, with appropriate steps to account for the additional degree of freedom of n, the parameter which describes the change in viscosity for shear changes:

$$\mu = \mu_0 |\frac{\partial u}{\partial z}|^{\frac{1}{n}-1} \tag{1}$$

where μ_0 is a constant and $\frac{\partial u}{\partial z}$ is the tangential velocity gradient in the fluid. Here $n = 1$ reduces the problem to the Newtonian case, as in *Kowal and Worster (2019b)*.

2 Details

The project began in mid-June shortly following the conclusion of the formative assessments in 2020. The first part of the project is to obtain the flux expressions in the non-Newtonian case. It is discovered that the integral to obtain the velocity expression did not have a closed form. It is therefore decided that the linearisation will be undergone prior to a transformation in variables illustrated in *Kowal and Worster (2019b)*.

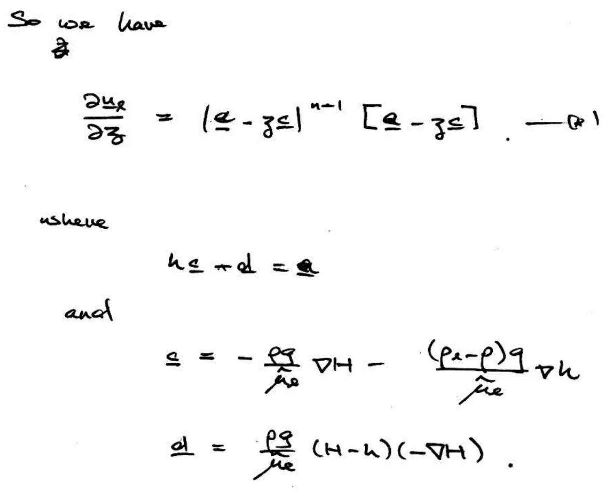

Figure 1: A section of the rough work in the initial part of the project. Here the partial differential equation does not have a closed form. Hence the expression on the right is first linearised and that allowed us to solve for a closed form and proceed.

After obtaining the flux expressions a similarity transformation was carried out on the expressions. Similarly the equation of state and boundary conditions of the problem is also similarly linearised and similarity transformed. The similarity transformation was done as follows:

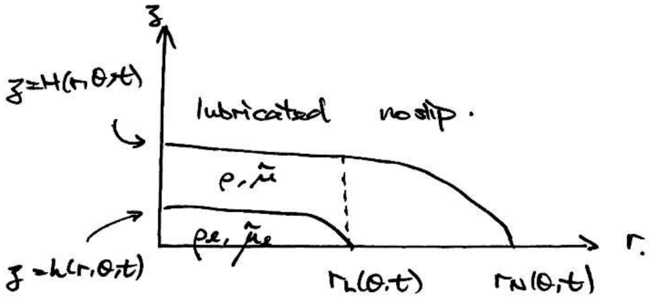

Figure 2: A sketched schematic of the problem. The problem is divided into two regions - the lubricated region and the no-slip region with different boundary conditions. Each region has an appropriate transformation rule.

The transformation rule for the lubricated region is

$$\begin{cases} \xi = \frac{r}{\xi_L(\theta,t)(\frac{g\rho}{\mu})^{\alpha} t^{\beta} Q_0^{\gamma}} \\ \phi = \theta \\ \tau = \log t \end{cases} \tag{2}$$

while for the no-slip region it is

$$\begin{cases} \xi = \frac{\frac{r}{\xi_L(\theta,t)(\frac{g\rho}{\mu})^{\alpha} t^{\beta} Q_0^{\gamma}} - 1}{\frac{\xi_N(\theta,t)}{\xi_L(\theta,t)} - 1} + 1 \\ \phi = \theta \\ \tau = \log t \end{cases} \tag{3}$$

Here α, β and γ are constants to eliminate the physical dimensions of the problem. The result is a problem reduced to the region $0 \leq \xi \leq 2$ with basic (zeroth order term) and perturbed (first order term) equations and their associated boundary conditions. Normal modes of the form $e^{i(kx+\sigma t)}$ is substituted to the perturbed equations to search for a normal mode solution of the problem. The equations are then solved numerically using *NDsolve* in Mathematica to obtain the height of the problem. The instability at the solution front of the lubricated region can be analysed by computing a residue matrix which gives an eigenvector with an associated eigenvalue, giving the residue between the calculated boundary condition flux and the flux calculated from the numerical solution. The effects of different parameters such as the viscosity constant ratio $\mathcal{M} = \frac{\mu_0}{\mu_{0,l}}$, normalised density difference $\mathcal{D} = \frac{\rho_l - \rho}{\rho}$ and parameter n on allowed unstable values of wavenumber k can then be investigated (quantities with the l subscript indicate that of the lower layer, those without the subscript refers to the upper layer).

3 Some results

Here is a $\sigma - k$ plot for parameters $\mathcal{M} = 5$, $\mathcal{D} = 1$ and $Q_0 = 0.1$ where Q_0 is the ratio between the inputting fluxes of the lower and upper layer.

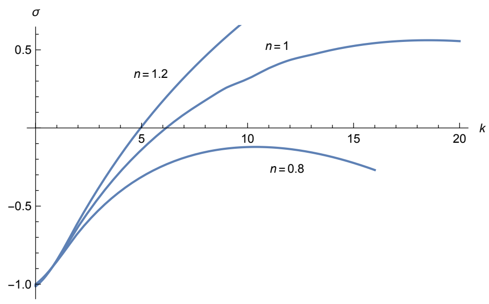

Figure 3: A graph that shows the σ and k relation. The problem is radially symmetric so only integer k values are allowed. The graph is obtained therefore by interpolation. Positive values of σ correspond to the development of instability for small perturbations around the lubrication front. From the diagram it can be deduced that decreasing n has a stabilising effect on the lubrication front.

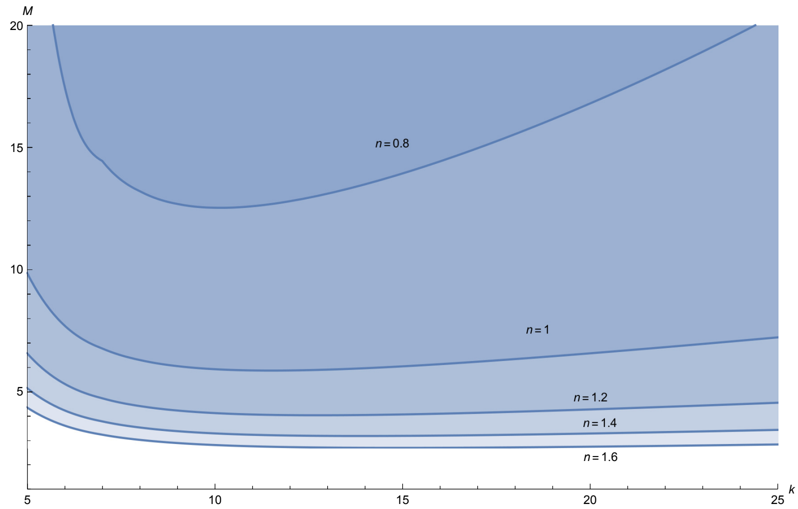

Figure 4: A map showing the regions of stable and unstable wavenumber regions of the problem with varying $\mathcal{M}$. The coloured regions are the unstable region. It can be generally shown that shear-thinning fluids ($n > 1$) increases the region for unstable wavenumbers.

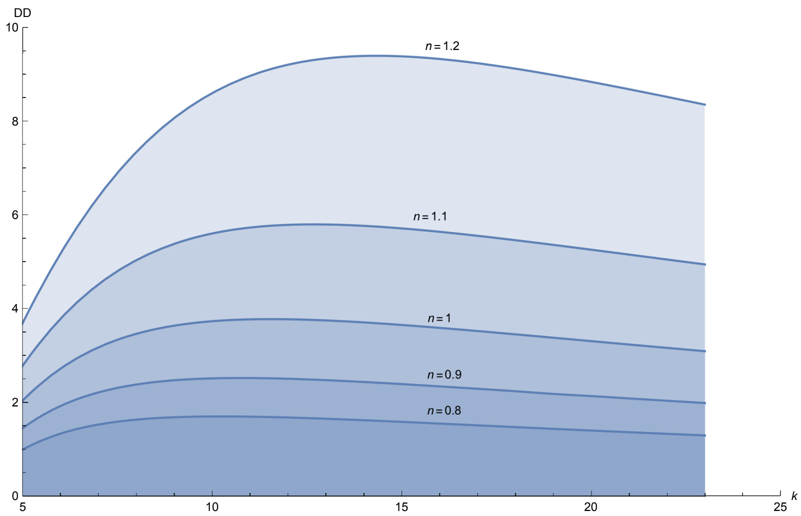

Figure 5: A map showing the regions of stable and unstable wavenumber regions of the problem with varying $\mathcal{D}$. The coloured regions are the unstable region. It can be generally shown that shear-thinning fluids ($n > 1$) increases the region for unstable wavenumbers.

The preliminary result from the project is that shear-thickening fluids $n < 1$ provides a stabilisation effect to any perturbation at the lubrication front while shear-thinning fluids $n > 1$ causes the instability to develop more easily (for more wavenumbers) at the lubrication front. The presence of a shear-thinning layer at the bottom of a liquid will, therefore, enhances the development of finger-like instabilities as fluids are injected from the source.

4 Conclusions

The effect of the parameter n on the instability of the lubrication front was investigated and it was shown that shear-thickening fluids tend to suppress the development of instability while shear-thinning fluids does the opposite. This was an interesting project and I am very pleased to have given the opportunity to work with and learn from a research from college. I am indebted to Dr Kowal for giving me this amazing opportunity and the college for financially supporting the project.

References

1. K. Kowal and M. Worster, *Stability of lubricated viscous gravity currents. Part 2. Global analysis and stabilisation by buoyancy forces, J.Fluid Mech. (2019), vol.871, pp. 1007-1027*